Intuitive Rules For School Geometry

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Abstract

We propose a system of rules for school geometry (Euclidean plane geometry), which is designed to be a basis from which to draw explanations of theorems for teaching purposes. The system describes a large enough portion of the foundations of geometry so that core theorems taught at school can be explained simply using deductive reasoning. A core strategy is to replace formal logic with intuitive rules when defining concepts. This strategy together with the natural language presentation enables nontechnical explanations that represent a balance between pure deductive reasoning and geometric intuition, which is appropriate for elementary education.

1 Introduction

The system of rules implements an approach whereby geometric concepts are not mathematically defined using mathematical logic and constructs such as sets. This contrasts particularly with axiomatizations of geometry (see, for example, [6], [3], [1], [7], and [2]), where logic is used as a basis for the existence of geometric concepts. In such axiomatic systems the aforementioned use of logic is characterised by a process of successive definition of new concepts starting with a limited number of initial concepts. The alternative strategy that we present avoids explicit definitions and merely lists intuitively true logical connections between various concepts that are not mathematically defined. Although the existence of the concepts is implicit in this way, the system allows for an explanation of the core theorems of school geometry, because there are enough intuitively true rules in the system to argue that the theorems taught in school hold. The use of logic is thus restricted so that it is used only to explain non-intuitive statements from intuitive statements.

Our implementation of this idea is not unique, but represents an instance of the aforementioned approach. It could be adapted to suit various audiences. In particular the choice of intuitive rules represents a basis and one could, for example, also declare that theorems about congruent triangles are intutively true rules, incorporating them into the system so that they do not require an explanation. Rules could be formulated by learners themselves in the context of discovery learning (see, for example, the discussion in [4]). The approach could also be adapted to teaching techniques that implement local organisations of geometry (as described in [5]). However the rules, as we presented them, represent a global system, and have been chosen as rules that are intuitively obvious given the explanations presented alongside the rules below.

The next section introduces the rules and includes explanations as to how the rules can be seen to be intuitively true. The subsequent section contains an analysis of the system, which includes an example of how the rules can be used to explain a theorem of school geometry. The section following this shows the extent to which the system characterises the Euclidean plane. Following this we summarise the idea and our analysis.

2 The system of intuitive rules

The method starts with a thought experiment whereby one imagines two points, the segment connecting the two points, the rays that are formed by extending the segment (one in each direction) and the line formed by extending in both directions. Then one imagines a sheet of paper and that the paper is extended in all directions to form a plane. Points, segments, rays, and lines are imagined as belonging to this plane. It is discussed that closed means that the endpoints of segments and the starting point of rays are part of the object whereas open means that they are not. Half-planes are also discussed with the appropriate interpretation of closed versus open. The thought experiment allows a learner to gain intuition about the Euclidean plane so that he or she perceives the following rules as true.

Object rules

1. We consider one object called plane and objects such as points, lines, open rays, closed rays, open segments, closed segments, open half-planes, closed half-planes, and angles that lie on the plane.
2. Every object, except the point itself, contains many points.

Line rules

1. There is exactly one line that passes through two given points.
2. Two distinct lines have at most one point of intersection and they are called parallel if they have no point of intersection. A line is said to be parallel to itself.
3. For every line there is a point that does not lie on the line.
4. Given a line and a point there is exactly one line parallel to the first line that passes through the point.

Remark 1 *It is suggested that the one initially asserts that the last rule is intuitively true, however that once the ideas have settled the possibility of non-Euclidean geometries is discussed, so that the rule is seen as one option among different possibilities equating to different geometries including non-Euclidean geometries.*

Segment rules:

1. An open segment connects two points called endpoints of the segment and lies on the line that passes through its endpoints.
2. There is exactly one open segment that connects two given points.
3. The endpoints of an open segment do not lie on the open segment.
4. A closed segment consists of an open segments and its two endpoints.
5. A point lies between two other points if it lies on the open segment that connects them.
6. Given three collinear points (meaning that they all lie on one line) exactly one of the three lies between the other two.

Ray and Half-plane rules:

1. A point on a line divides the line into two open rays called sides of the point on the line such that two other points of the line lie in different rays exactly when the initial point lies between them.

2. Every open ray is a side of a particular point on a particular line. A closed ray consists of an open ray and its starting point.

3. A line divides the plane into two open half-planes called sides of the line such that an open segment intersects the line exactly when its endpoints lie in different half-planes.

4. Every open half-plane is the side of a unique line. A closed half-plane consists of an open half-plane and the line it is a side of.

Angle rules:

1. An angle contains a point called the vertex and two distinct open rays starting at the vertex called legs.

2. For two distinct open rays with the same starting point there are two angles that have these rays as legs. A further distinct open ray with the same starting point lies on exactly one of these angles.

Length rules:

1. Every segment has a unique length.

2. Given an open ray and a length there is a unique point on the ray whose distance to the starting point of the ray equals the given length.

3. If point B lies between points A and C, then the distance between A and C is the sum of the two distances between A and B and between B and C, and the distance between A and C is longer than either of the distances between A and B and between B and C.

4. Given an open segment and a natural number n larger than 1, one can find $n-1$ points on the open segment that divide it into n open segments that each have the same length.

5. The sum of the lengths of two sides of a triangle is longer than the length of the third side.

Angle size rules:

1. Every angle has a unique size.
2. Given an open ray and an angle size there are exactly two angles with the given size that have the open ray as a leg.
3. If an angle consists of two angles that share a vertex and a leg, but otherwise have no points in common, then the sum of the angle sizes of the two angles equals the size of the original angle. The size of each of the two angles is smaller than the original angle.
4. Every angle can be divided into two angles of equal size that only share a vertex and a leg. The shared leg is called the angle bisector.

Polygons and area rules:

1. A polygon consists of $n > 2$ distinct points called vertices and n open segments called edges, each of which connects two of the vertices. Each vertex is an endpoint of exactly two edges and none of the edges intersect.
2. If two polygons share one edge and the two vertices of the edge, but are otherwise on different sides of the line that passes through the endpoints of the shared edge, then the sum of the areas of the two polygons equals the area of the polygon consisting of all the vertices and edges of both polygons except for the shared edge. The area of each polygon is smaller than the area of the new polygon.

Remark 2 *The aforementioned thought experiment can be extended so that the learner perceives the next rules about motion as true. One can rotate, translate, and flip over the paper in order to explain that the plane does not inherently change, but that the objects assume new positions. In particular it should be clear that, for example, the lengths of a segment equals the length of the segment in its new position once it is translated, rotated, or reflected.*

Motion rules:

1. A translation, a rotation, or a reflection of the plane relates each object with another object of the same type, which is called a translation, rotation, or reflection of the original object. All other relations involving the original objects hold when referring to the moved objects including length of segments, size of angles, and area of polygons.

2. Given two distinct ordered points there is exactly one translation such that the first point lands on the second point.

3. No point remains in the same place if it is translated.

4. A rotation has one fixed point. It is referred to as a rotation about this point.

5. Given two distinct ordered open rays with the same starting point, there is one rotation about the starting point so that the first ray lands on the second ray.

   Note: It should be explained that the rotation about a point in a particular direction of rotation is regarded as the same rotation about the point as the rotation in the opposite direction of rotation through an angle size that is 360° minus the original angle size . An alternative would be to make a distinction between the direction of rotation, but this carries with it the disadvantage that the group of motions is later harder to define.

6. After a translation or rotation one can translate or rotate the plane back to its original position.

7. A reflection about a line of a plane leaves the points on the line fixed, but a point of the plane that is not on the line is reflected onto the other side (half-plane) of the line.

8. After reflecting twice all objects land back at their starting positions.

Archimedean rule

Given a point, a translation, and a length, successive translation of the point will result in a new point whose distance from the first point is further than the given length.

3 An analysis of the rules

As already mentioned, the rules describe intuitively true logical connections amongst geometric concepts, but they do not define these concepts explicitly. Some of the concepts are geometric objects and some are concepts such as length, angle size, area, and motion, which do not correspond to geometric objects. The idea of such a system is to exploit intuitive connections between many different concepts, fitting them logically together, so that the overall logic that governs, and which could be used to define the Euclidean plane, is described through the totality of all of the intuitively true rules. One should note that the principle of not making mathematical definitions extends to the Euclidean plane itself, because the Euclidean plane is also only mentioned by the rules, but it is not defined by them.
A characteristic of the system is that it avoids mention of, for example, all of the points of a segment or all of the rotations about a given point. Rather the system is designed to allow deductive reasoning starting from hypotheses that are finite in nature. This characteristic reflects the essentially finite nature of school geometry, where figures consist of a finite number of key points, segments, lines and so forth. As an example of this role of finiteness we consider an explanation of the following theorem, a standard theorem taken here as an example, whose hypotheses refer to two triangles, each consisting of three points (the verticies), three open segments (the edges), and three interior angles so that the theorem is essentially finite in nature even if there are infinitely many points on each edge.

Theorem 1 *(ASA Congruence) If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\alpha = \delta$, $b = e$ and $c = f$, then $\triangle ABC \equiv \triangle DEF$.*

Remark 3 *We use the convention that verticies are denoted using the uppercase latin alphabet, that an edge is denoted by the letter of the lowercase latin alphabet corresponding to uppercase letter used to denote the vertex opposite it, and that an interior angle at a vertex is denoted by the letter of the lowercase greek alphabet corresponding to the uppercase letter used to denote the vertex.*

Explanation: If $A \neq D$ then one can translate so that A lands on D, because of the second rule of motion. Thus there is a possibly translated triangle $\triangle D'E'F'$ such that $D' = A$. If now the rays $\overrightarrow{AB}$ and $\overrightarrow{D'E'}$ are not the same, one can rotate about $D' = A$ so that $\overrightarrow{D'E'}$ lands on $\overrightarrow{AB}$, because of the fifth rule of motion. Thus one arrives at a possibly translated, possibly rotated $\triangle D''E''F''$ with the property that the rays $\overrightarrow{AB}$ and $\overrightarrow{D''E''}$ are the same. One now checks whether the point F'' is on the same side of the line $AB = D''E''$ as C and if not, then we reflect along AB. Thus we arive at a possibly translated, possibly rotated, and possibly reflected triangle $\triangle D^*E^*F^*$, such that $\overrightarrow{AB} = \overrightarrow{D^*E^*}$ and such that C and F^* are on the same side of $AB = D^*E^*$. The first rule of motion implies that each of lengths of the edges and size of the interior angles of $\triangle D^*E^*F^*$ is the same as corresponding length or interior angle of $\triangle DEF$. Therefore it follows from $c = f$ and the second rule about lengths that $E^* = B$. Further it follows from the second rule about angle sizes that $\overrightarrow{AC} = \overrightarrow{D^*F^*}$ so that a further application of the second rule about lengths implies that $F^* = C$, because $b = e$. Thus $\triangle DEF$ lands on $\triangle ABC$, so that corresponding edges have the same length and corresponding interior angles have the same size.

In the above explanation the finite set of concepts mentioned in the hypotheses are expanded. For example, if A and D are in fact different points, the first step is to use the rules of motion to conclude that there is a translation, whereby D lands on A. In this way the hypotheses are expanded to include the existence of this translation, but remain finite in nature. The translation is then used in the next steps of the explanation, which further expand the hypotheses in a finite way until enough information is available so that the conclusion follows.
The explanation is not completely rigorous, because the proof draws on intuition regarding what is meant by the interior angles of a triangle. However, once $\triangle DEF$ has landed on $\triangle ABC$ the conclusion that the interior angles correspond is intuitively clear. This reflects that the system has been designed to allow deductive reasoning to be used as a tool of explanation, but where a proof based on pure deduction through formal logic is not the goal. As such the system represents a teaching philosophy that is a compromise between pure logic and intuitive reasoning, which is suitable for elementary education.
Logical independence and consistency are important characteristics of axiomatic systems. However, as the concepts are not defined, but merely governed by the proposed intuitive rules, it is not necessary to analyse these characteristics. This simplification is appropriate for elementary education,

because the system promotes the use of logic, but only insofar as to prove statements from intuitively true statements, making the idea of logic and of proof easier to understand.

4 The extent of the rules

An important question is whether the rules categorise the Euclidean plane accurately. The answer lies in the following theorem, which indicates that the rules can be used to argue any statement of school geometry provided that a proof of the statement does not depend on the completeness of the Euclidean plane.

Theorem 2 *We consider a set E and collections of subsets, one collection for each of the types of geometric object as mentioned in the rules except in the cases of points and of the plane itself. We consider the rules as logical statements, which, when referring to geometric objects, refer to a member of the collection corresponding to that particular type geometric object, except in the case of points where it refers to a member of the set and in the case of the plane where it refers to the whole set. If the set E is complete with respect to the naturally arising metrics, then it is equivalent as a metric space to $\mathbf{R}^2$, and the concepts correspond to their standard definitions in the metric space context.*

In order to prove the theorem we consider how a real number can be associated to a length using the rules. The way to do this is with the help of a unit length and through the application of the length rules and the Archimedean rule. This standard technique is to successively translate a point using a translation corresponding to the unit length and to count the number of translations until a further translation would result in the point being further from its starting position than the length one wishes to measure. This yields the whole portion of the real number corresponding to the length as measured by the unit. The full decimal expansion of the number is found by applying the same technique successively to the remaining length measuring with one tenth of the unit length (the existence of one tenth of the length is given by the fourth length rule). This yields a decimal representation of a real number and the real number is subsequently associated to the length in question. The motion rules guarantee that the real number does not depend on the choice of point or translation, making the definition well-defined. The association of the real numbers to lengths in this way is consistent with the third length rule about the sum of two lengths and the concepts of longer. The triangle inequality rule (fifth length rule) implies that the distance between two points as measured by a particular unit is a metric on the set E. With this metric E becomes a metric space.

Remark 4 *A similar technique could be used to associate a degree to each angle size in a manner consistent with the angle size rules. One would use the angle bisector instead of the fourth rule for lengths that allows a length to be divided into ten equal lengths. Furthermore, one can associate the area of a rectangle with the product of its length and breadth (each being real numbers measured by the same unit). This association is consistent with the rules governing the sum of areas (second rule about polygons and area). Furthermore, the theorem of Pythagoras follows from standard arguments based on area considerations.*

The task is to create a metric-preserving bijection between E and $\mathbf{R}^2$. Ones takes two points in E, whose existence follows from the object rules. We then consider the line through these points and a point not on this line, whose existences follow from the line rules. We then reflect the point about the line so that the point not on the line is reflected onto the other side of the line (see rules of motion). The line through the point and its reflection is perpendicular, because the size of angles are preserved by reflection. so that we have perpendicular axes. We choose a positive and a negative side of the origin for each axis and assign a real number to each point on the axes by measuring its distance to the origin using a metric defined by a particular unit length (as described above). We assign coordinates to each point on E by reflecting the point along each axis and taking each coordinate as the real number corresponding to the number associated with the intersection point with the corresponding axis. In this way we define a map from E to $\mathbf{R}^2$. The map is well defined because of the uniqueness of perpendiculars (a characteristic that follows easily from the rules). The map is injective, because if two dinstinct points in E map to the same coordinates in $\mathbf{R}^2$ then this would correspond to two rectangles with the same length and breadth, with one corner being the origin and whose opposite corners are different but in the same quadrant. This cannot be because of the uniqueness of perpendiculars and the uniqueness of intersection of lines (second rule about lines). The map is surjective, because the completeness of E yields a full set of real numbers on each axis, so that a point in E can be found corresponding to each coordinate of $\mathbf{R}^2$ using the appropriate limiting process. The metric is preserved by the map, because of the theorem of Pythagoras, so that it follows that the metric spaces are equivalent. That each of the concepts mentioned in the rules correspond through the map to the concept as per its usual definition in $\mathbf{R}^2$ follows from standard analysis.

5 Summary

The use of intuitively true rules as a starting point for explanations of the theorems of school geometry allows an argument to be both simple and, to a large extent, mathematically rigorous. The reason is that the system has enough logical rules to categorise the Euclidean plane up to analytic completeness. This allows one to teach Euclidean geometry by referring to geometric concepts, but without the technicalities involved in defining these concepts using mathematical logic and mathematical constructs such as sets.

References

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