IS PROBLEM POSING ABOUT POSING “PROBLEMS”? A TERMINOLOGICAL FRAMEWORK FOR RESEARCHING PROBLEM POSING AND PROBLEM SOLVING

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In this literature review, we critically compare different problem-posing situations used in research studies. This review reveals that the term “problem posing” is used for many different situations that differ substantially from each other. For some situations, it is debatable whether they provoke a posing activity at all. For other situations, we propose a terminological differentiation between posing routine tasks and posing non-routine problems. To reinforce our terminological specification and to empirically verify our theoretical considerations, we conducted some task-based interviews with students.

INTRODUCTION

In his article *“The heart of mathematics”*, the mathematician Paul Halmos concluded: “I do believe that problems are the heart of mathematics, and I hope that as teachers, in the classroom […] we will train our students to be better problem-posers and problem-solvers than we are” (1980, p. 524). He emphasizes two activities: The first activity, problem *solving*, received a lot of attention in the last decades of mathematics education research, especially since Pólya’s (1945/1973) and Schoenfeld’s (1985) seminal works. The second activity, problem *posing*, has also been emphasized as an important mathematical activity by many mathematicians (Cantor, 1932/1966; Lang, 1989) and mathematics educators (Brown & Walter, 1983; English, 1997; Kilpatrick, 1987; Silver, 1994). As an important companion of problem solving, it can encourage flexible thinking, improve problem-solving skills, and sharpen learners’ understanding of mathematical content (English, 1997).

However, problem posing hast not been in the focus of mathematics education research (Cai, Hwang, Jiang, & Silber, 2015). As a result, “the field of problem posing is still very diverse and lacks definition and structure” (Singer, Ellerton, & Cai, 2013, p. 4). Therefore, we conducted a literature review and critically compared different problem-posing situations used in research studies and theoretical papers that were labeled as *problem posing*.

This review revealed that one aspect in which this field of research lacks definition and structure is the variety of situations in empirical studies that are labeled as *problem posing*. This can be exemplified by two studies. Both Cai and Hwang (2002) as well as Arıkan and Ünal (2015) show strong links between students’ problem-posing and problem-solving performance. However, a

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comparison of the problem-posing situations used in both studies reveals significant differences. Cai and Hwang (2002) (see Table 1, Situation 1) invite students to pose at least three tasks with varying degrees of difficulty for a given dot pattern. This leads to a creative activity of exploring different paths to pose numerous tasks, such as “How many white dots are there in the twentieth figure?”, or “How many black dots are there in the first five figures?” (p. 413). Arıkan and Ünal (2015, see Table 1, Situation 2) provide a defined calculation and three tasks for which only one applies to the given calculation. Therefore, there is only one correct answer and the situation does not encourage posing further tasks. Thus, if both studies find a strong link between problem solving and problem posing, they do not refer to the same relationship, because the situations they use differ significantly from each other. This is like comparing a non-routine problem with an algorithmic word problem and calling both *mathematical problem solving*.

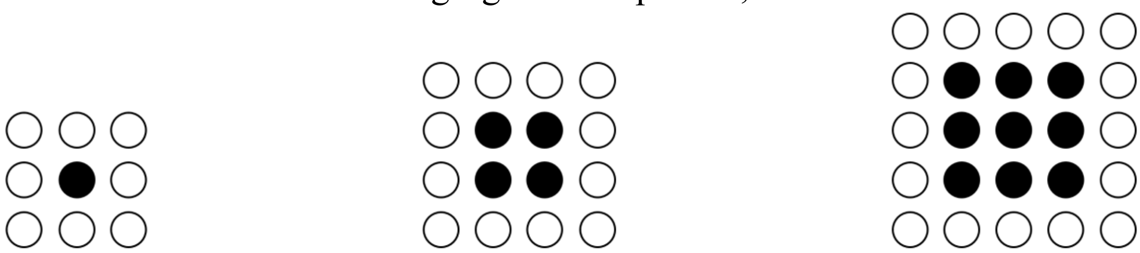
	Situation
1	Mr. Miller drew the following figures in a pattern, as shown below.  (Figure 1) (Figure 2) (Figure 3) For his student’s homework, he wanted to make up three problems based on the above situation: an easy problem, a moderate problem, and a difficult problem. These problems can be solved using the information in the situation. Help Mr. Miller make up three problems and write these problems in the space below. Cai & Hwang (2002, p. 405)
2	Which one of the below problems can be matched with the operation of 213 + 167 = 380? A) Osman picked up 213 pieces of walnut. Recep picked up 167 more pieces of nuts more than Osman. What is the total amount of the nuts that both Osman and Recep picked up? B) On Saturday, 213 and on Sunday 167 bottles of water were sold in a market. What is the total number of bottles of water that were sold at this market on these two days? C) Erdem has 213 Turkish lira. His brother has 167 lira less than that. What is the total amount of money that both Erdem and his brother have? Arıkan & Ünal (2015, p. 1410)

Table 1. Problem-posing situations from studies with similar research questions.

This striking difference between two studies, which are based on a similar research question, motivated a more in-depth investigation of problem-posing situations used in research studies. Apparently, the term *problem posing* is used for a variety of situations that seem to have characteristic differences. Therefore, we propose a terminology to be able to differentiate between those situations and to prevent misinterpretations of research results.

A similar approach – clarifying terms – has been made in research on problem solving, decades ago. The term *problem* has been and still is used in multiple and often contradictory meanings, which makes it difficult to interpret the literature. In some cases, the term *problem* is used for any kind of mathematical task without differentiating between *routine tasks* or *textbook exercises* and *non-routine problems*, as researchers like Schoenfeld (1985, 1992) suggest.

THEORETICAL BACKGROUND

In this chapter, we want to present the current understanding of the term *problem posing*. Afterwards, it will be further analyzed by breaking it down into its etymological components, *problem* and *posing*.

What is *problem posing*?

There are two definitions of the term *problem posing*, at least one of which is used or referred to in almost all mathematics education research papers on the topic. The first definition was stated by Silver (1994, p. 19), who defines problem posing as the activity of generating new problems and reformulating given problems which, consequently, can occur *before*, *during*, or *after* a problem-solving process. The second definition comes from Stoyanova and Ellerton (1996, p. 518), who refer to problem posing as the "process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems". Both concepts of problem posing are not very restrictive and can be applied to a wide spectrum of situations. In the following, we adopt the definition of Stoyanova and Ellerton (1996) as the underlying understanding within this paper.

Stoyanova and Ellerton (1996) differentiate problem-posing situations between *free*, *semi-structured*, and *structured* problem-posing situations, depending on their degree of structure. Free situations provoke the activity of posing problems out of a given, naturalistic or constructed situation without any restrictions. In a semi-structured situation, the problem poser is invited to explore the structure of an open situation by using mathematical knowledge, skills, and concepts of previous mathematical experiences. It is noticeable that the differentiation between *free* and *semi-structured* situations is difficult because there is no sharp demarcation. We, therefore, plead to merge free and semi-structured situations, resulting in *unstructured* situations which have varying degree of restrictions. In structured situations, people are asked to pose further problems based on a specific problem, e.g. by varying its conditions.

What is *posing*?

We now fragment the term *problem posing* into its two components, starting with the definition of *posing*, for which we consulted dictionaries. The Cambridge Dictionary (2018) defines "to pose" as "to ask a question".

What is a *problem*?

In mathematics education research, the term *problem* has been (and still is) used for any kind of mathematical task, leading to some difficulties and misinterpretations in reading the research literature (Schoenfeld, 1992, p. 337). Therefore, researchers like Schoenfeld (1985) suggest to differentiate between mathematical tasks that are *routine tasks* or *exercises* and *non-routine problems*. Following Schoenfeld (1985), we consider *(mathematical) tasks* to be the overarching category that can be further differentiated into *routine tasks* or *exercises* "if one has ready access to a solution schema" (p. 74) and *non-routine problems* if the individual has no access to a solution schema.

In most cases, the decision whether a task is a routine task or a non-routine problem is evident. Nevertheless, this attribution is specific to the individual; a problem which is a non-routine problem for one person can be a routine task for another person who knows a solution scheme (Dörner, 1979; Rott, 2012; Schoenfeld, 1992). Thus, the demarcation between these categories may not be sharp but the extreme cases are clearly recognizable (Pólya, 1966, p. 126).

METHODOLOGY

We systematically gathered problem-posing situations from 185 empirical studies and theoretical papers about problem posing from the A*- and A-ranked journals in mathematics education research (as classified by Törner & Azarello, 2012, p. 53), the Web of Science, papers from the PME, and papers of the collection of Singer, Ellerton, and Cai (2015) as well as the collection of Felmer, Pehkonen, and Kilpatrick (2016). With the situations resulting from this review, we have conducted a qualitative content analysis. An inductive category formation with regard to the term *posing* lead to two categories, which will be presented in the analysis. A deductive category assignment with regard to the term *problem* lead to another two categories, which will be presented in the analysis.

In a small empirical study ($n = 4$ participants), we tested these theoretical considerations by conducting task-based interviews with situations from the review. The answers were analyzed with regard to the theoretical considerations of the analysis presented in this article.

TERMINOLOGICAL ANALYSIS

Is it *posing*?

There are some situations for which it has to be discussed whether the term *posing* fits to the activity it is supposed to induce (see Table 2).

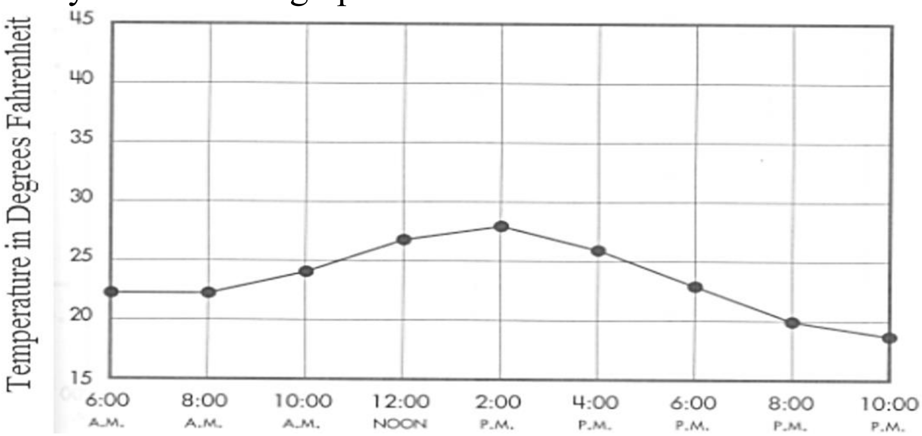
	Situation
3	Write a story to match the graph shown below.  Jiang & Cai (2014, p. 397)
4	Write a question to the following story so that the answer to the problem is '385 pencils'. 'Alex has 180 pencils while Chris has 25 pencils more than Alex'. Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman (2005, p. 152)

Table 2. Problem-posing situations discussed regarding the term *posing*.

Situation 3 invites providing a context to a given data or calculation. A graph showing the temperature of an unknown place from 6:00 a.m. to 10:00 p.m. is given; the task is to write a story that matches with this graph. We conclude that this task does not necessarily lead to a question that needs to be solved afterwards.

Our empirical study confirmed this theoretical conclusion: three out of four participants did not ask a question when working on this situation. In our view, this situation, therefore, is no *posing* activity. Nevertheless, it is an important activity for students and, thus, of interest in mathematics education research – but not in research on problem posing. We refer to these situations as *context providing tasks*.

In situation 4, students are invited to search for the question to a given context and its answer. The sought-after task, however, is predefined: "How many pencils do they have in total?" Reacting to situation 4 in the expected way leads to posing a task, but once the right question for the situation has been found, there is no task that can be worked on because it has already been solved. Furthermore, because the situation offers a defined goal, searching for the question that matches the predetermined situation and answer makes working on this situation equivalent to solving a *reversed task*. These are characterized as mathematical tasks with a defined goal and an undefined question (e.g., Bruder, 2000). This applies to situation 4 and therefore we consider describing these and similar situations with a given goal and a basically unambiguous sought-after question not as *problem posing* but rather as a *reversed task*.

Both context providing tasks and reversed tasks have in common that they have a solution for which it can be decided whether it is correct or incorrect. Therefore, we summarize them under the term *answering tasks*. *Answering tasks* are to be distinguished from *problem-posing situations*. Whether a given setting is an answering task or a problem-posing situation can be determined a priori by analyzing whether the particular characteristics presented above apply.

Is it a *problem*?

In the following, we want to use the differentiation of mathematical tasks into routine tasks and non-routine problems to further differentiate problem-posing situations. In Table 3, there are four situations sorted by the presented and adapted categories of Stoyanova and Ellerton (1996).

The shopping shelf in situation 5 comes along with three routine tasks for which additional questions are to be posed. As the given tasks are routine tasks, the situation provokes posing routine tasks like "You want to buy two badminton rackets, a football and a basketball and you have $200 in your pocket. Do you have enough money for these products? If not, what is the difference?" In fact, six out of six tasks posed by the participants in the empirical investigation were routine tasks.

Situation 6 is quite similar to situation 5 by also stating a structured situation with a task to be solved. However, in contrast thereto, a non-routine problem is given. Tasks should be posed by constraint or goal manipulation which, consequently, leads to further non-routine problems like: "Which radius should the smallest circle have, so that the area of the largest circle is π?" The empirical investigation confirmed this evaluation: eight out of eight tasks posed by the participants were non-routine problems.

Comparing these two situations, situation 5 provokes posing routine tasks whereas situation 6 provokes posing non-routine problems. We want to apply this established differentiation of mathematical tasks between routine tasks and non-routine problems on the structured problem-posing situations 5 and 6. This paper introduces the terms *routine task posing* and *non-routine problem posing*. The former refers to the process of posing routine tasks, and the latter refers to the process of posing non-routine problems. However, it is not sufficient to assess the initial task of structured situations in terms of whether it is a non-routine problem or a routine task. Even initial routine tasks can lead to a mathematically rich non-routine problem. Therefore, when labeling a situation as *routine task posing* or *non-routine problem posing*, it is necessary not only to assess a priori the initial task of a structured situation but also to assess a posteriori the emerging tasks.

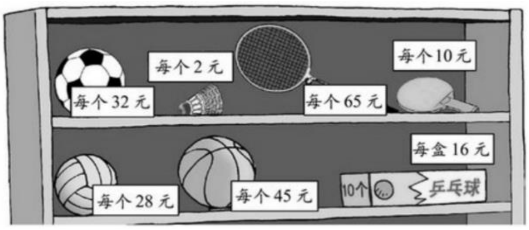
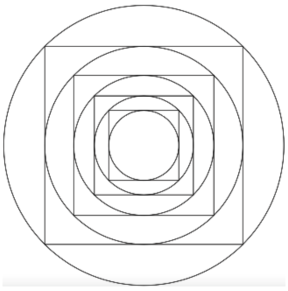
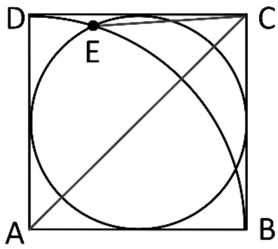
		Situation
structured	**5**	(1) If we want to buy 5 volleyballs, how much do we need to pay? (2) If we bought three footballs, and paid the cashier 100 dollars, how much can we get for change?  (3) If I want to buy one badminton racket and 10 badminton shuttlecocks, how much do I need to pay? (4) Please pose two more questions and answer them. Jiang & Cai (2014, p. 396)
	6	 For the figure on the left, one mathematics problem we could ask is: *Given that the radius of the smallest circle is one unit, what is the ratio of the area of the largest circle to the area of the smallest circle?* 1. Think about how to solve this problem. […] 2. Pose problems using constraint manipulation or goal manipulation strategy according to the given figure, or the problems you have posed, or any other ideas you have. […] Xie & Masingila (2017, p. 116)
unstructured	**7**	Write a problem based on the following picture:  Christou et al. (2005, p. 152)
	8	The figure contains: the square *ABCD*, the circle inscribed in this square, and the circular arc of centre *A* and radius *AB*. Pose as many problems as possible related to this figure […]  Singer, Voica, & Pelczer (2017, p. 39)

Table 3. Problem-posing situations discussed regarding the term *problem*.

This qualitative difference between posing routine tasks and posing non-routine problems also occurs in the two unstructured situations 7 and 8. The former situation provokes routine tasks like: “For their kitchen, a family needs a stove with an oven and a fridge. What do they have to pay in total?” Of course, also non-routine problems can be posed, but they are less likely in this situation. This is reinforced by the empirical investigation in which four out of four tasks posed by the participants were routine tasks. Situation 8 induces posing tasks like: “What is the ratio of the line segments *AC* to *CE*?” or “What is the area of the rounded segment of the inscribed circle?” These are non-routine problems which the figure provokes to pose. Nonetheless, there are also routine tasks that could occur, though they are far less interesting with regard to the information the situation provides. Actually, 10 out of 16 tasks posed by the participants in the empirical investigation were non-routine problems.

We now apply the new terms: situation 7 provokes the activity of *routine task posing* and situation 8 provokes the activity of *non-routine problem posing*. However, for the unstructured situations, this distinction is less pronounced than for the structured situations 5 and 6. Since no tasks are predefined, you are not immediately urged in the direction of a specific type of task in the process of posing. Depending on association, motivation, and mathematical experience, both routine tasks and non-routine problems can be posed. Because of these difficulties to predict whether the given situation provokes posing routine tasks or non-routine problems, the differentiation between *routine task posing* and *non-routine problem posing* is not dichotomous but rather a continuum.

CONCLUSION

The stated situations and their characteristics reveal two aspects. First of all, the discussed *answering tasks*, which consist of *context providing tasks* and *reversed tasks*, do not fit into the stated understanding of *posing*. We want to distinguish them from *problem-posing situations*. Whether a situation or task is an *answering task* or a *problem-posing situation* can be assigned a priori by analyzing the characteristics of the situation or task. Secondly and similar to the terminological differentiation of mathematical tasks into *routine tasks* and *non-routine problems* (Pólya, 1966; Schoenfeld, 1985), we want to differentiate between *routine task posing* and *non-routine problem posing*. An a priori assignment of this differentiation is without complete certainty since it also depends on the problem poser whether a situation provokes routine task posing or non-routine problem posing. Instead, it is necessary to attribute a posteriori – and for each task individually – whether the posed tasks are routine tasks or non-routine problems. Since the problem-solving research has benefited from the differentiation between *routine tasks* and *non-routine problems* (Schoenfeld, 1992), it is assumed that it could also be beneficial for problem-posing research.

The framework can now be used to determine the differences between the studies from the introduction (see Table 1) more precisely. While Cai and Hwang (2002) used problem-posing situations that supposedly induce non-routine problem posing, Arıkan and Ünal (2015) used reversed tasks which – on the basis of this framework – we do not consider to be a problem-posing activity. As mentioned in the introduction, the field of problem posing lacks definition and structure. This paper's framework is a contributive attempt to close this gap.

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