CHALLENGING TASKS – 13 YEARS ZINGY-MATHEMATICS-OLYMPIAD

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Abstract
The Zingy-Math-Olympiad [ZMO] was held over 13 years with a total of 2102 third graders (~49.43% girls) in the context of university seminars on mathematical giftedness. The almost achieved gender parity is ensured by the requirement that one girl and one boy per participating class should be sent to the ZMO as their mathematics representatives. The tasks, which vary in their degree of difficulty, are divided into 7 rubrics: R1 simple arithmetical introductory tasks, R2 & R3 more demanding arithmetical skills tasks, R4 combinatorically solvable tasks, R5 word problems, R6 pattern and geometric figure tasks, R7 final tasks. The students' performance in the seminar consists in developing task sets and the evaluation with final scoring of the task completions, all based on research and discussions on relevant literature. In accordance with previous research, there is a tendency for the participating boys to be (slightly) superior to the participating girls, especially in the peak performance. In addition to an analysis of the tasks and their processing, a key question for the future is which cognitive abilities are influencing factors for successful mathematical problem solving in order to be able to address them specifically in mathematics lessons in a further step. So far, there are only meager approaches to this.*)

Keywords: *functional-/predicative-logical thinking, gender differences, mathematical giftedness, primary school, problem solving*

ZMO: ZINGY-MATHEMATICS-OLYMPIAD

In the city and district of Osnabrück, for 13 years the ZMO was conducted as a Mathematics Olympiad for third graders as part of seminars on mathematical talent, primarily with students in elementary school teaching. In order to address girls as well as boys and to involve a large number of the over 100 elementary schools in the region, it was decided as a condition of competition that one girl and one boy per participating class should be sent as their math representatives. Each class had to submit a creative math project that could be designed in many ways to fit the theme of the respective round (e.g. fairy tales, outer space, sports, the Middle Ages, underwater worlds), including crafts, books, games and theater plays. All creative contributions were digitally processed and presented to the public on the Internet. The best contributions were awarded, and children and teachers were invited to an award ceremony at the University of Osnabrück. The math representatives of the classes met in groups of about 20 on a Saturday morning in the second half of the school year. During the first years, the meeting was at the elementary school hosting the Mathematics Olympiad. In the later years – due to the high number of participants – meetings took place at the University of Osnabrück. During the meetings, the math representatives solved math problems specifically created for each Mathematics Olympiad while university students monitored the process and ensured the competitive conditions. Initially, persuasion was necessary to explain that girls and boys alike are equally capable of taking part in a Mathematics Olympiad – instead of one girl and one boy, teachers from their classes recommended two boys or one single boy – but in the long run, no more discussions were necessary. Over the years, a total of 1039 girls and 1063 boys have participated, i.e. a total of N=2102 third graders (~ 49.43% girls). All

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children received a certificate for their achievements in completing the mathematical tasks, with increasing scores divided into bronze (~40% of the children), silver (~40% of the children) and gold (~20% of the children). The diamond category for 3rd, 2nd, 1st place for both girls and boys was awarded for special success. In addition to the diamond certificate, the first-placed girl and boy received one of the two ZMO challenge cups during the award ceremony with teachers and family members.

Two team members corrected and scored each completion of a task on the basis of a point distribution determined by the team. A team of experts consisting of research assistants, experienced students and the scientific management of the ZMO intensively reviewed and discussed the task completions of those girls and boys whose points were among the top 30% of the points achieved until the scoring was finalized.

The highest scores in each of the 13 years were achieved by 10 boys and 3 girls. Moreover, fewer girls than boys made it into the gold group. In the diamond group, the achieved points of the girls were on average around 92.5% of those of the boys, in the top 20% 93.6%, respectively. Given the social focus on and tendency to admire the very best, there is certainly still a need for educational work and a more comprehensive investigation of the reasons. It may be necessary to reconsider the ranking of places, which suggests greater differences than the associated differences in performance (measured in point differences).

Table 1 gives a first impression of the differing performances of boys and girls in the seven different task rubrics. R1 are simple arithmetic entry tasks, R2 & R3 are more demanding arithmetic skills tasks, R4 are combinatorically solvable tasks, R5 are word problems, R6 are pattern and geometric figure tasks, R7 are final tasks. Examples are presented in the following section.

Table 1. Balance from 13 years: Point mean values [P-MV] by gender in the rubrics R1-R7 and respective percentage of girls' P-MV.

Rubric	Boys‘ P-MV	Girls‘ P-MV	Percentage Girls‘ P-MV
R1	29,08	29,26	100,60
R2	25,32	21,24	83,88
R3	23,34	22,28	95,47
R4	19,27	20,84	108,15
R5	24,27	20,68	85,21
R6	18,52	18,62	100,54
R7	17,38	17,84	102,62

Statistical evaluations using SPSS 26 substantiate the differences with regard to rubrics 2, 4 and 5. In the following, the results are given with regard to:

- *C-GB*: Comparison of all girls & boys across all rubrics and all years
- *C-GB-20*: Comparison of best 20% each across all rubrics and all years
- *C-GB-R*: Comparison of all girls and boys to rubrics across all years
- *C-GB-R-20*: Comparison of best 20% each to rubrics across all years

Analyses with the Kolmogorov-Smirnov and Shapiro-Wilk test revealed that the individual scores per task (1-10) as well as the individual total scores are not normally distributed ($p < .001$ for all variables). Therefore, Mann-Whitney U-test was calculated for the comparisons above.

- *C-GB*: In terms of total score, there was no significant difference between girls and boys in the overall sample ($p = .171$, $M_{girls} = 32.88$, $SD=11.41$, $M_{boys} = 33.85$, $SD = 12.43$). However, the large SD value is noticeable.

- *C-GB-20*: On the other hand, a significant difference could be found for the best 20% ($p < .001$, M_{girls} =49.80, SD=5.89, boys =52.80, SD = 5.61).
- *C-GB-R*: Significant differences were found in rubrics 2, 4 and 5
  (rubric 2: $p < .001$, $M_{girls} = 2.77$, SD=2.63, $M_{boys} = 3.22$, SD=2.77,
  rubric 4: $p < .05$, $M_{girls} = 3.73$, SD=1.85, $M_{boys} = 3.49$, SD = 2.05,
  rubric 5: $p < .05$, $M_{girls} = 1.99$, SD=2.24, $M_{boys} = 2.56$, SD = 2.13).
- *C-GB-R-20*: Significant differences can be seen in rubric 5 ($p < .05$, $M_{girl} = 4.65$, SD = 2.24, $M_{boys} = 5.15$, SD = 2.11). The striking feature of this rubric 5, the world problems, is remarkable.

The results caused early discussions. Girls can be superior to boys in mathematics as well as the other way around, but the proportions are not balanced. This poses a challenge for mathematics lessons in elementary school: How can children, especially girls, be even supported more thoroughly in the development of their mathematical thinking, especially their arithmetical thinking? Here, the ability to think means being able to independently come up with an idea and not just to apply what they have learned. Impairments make attempts to solve problems more difficult.

In view of the results, there has been constant reflection on how the gap in the performance of girls and boys could possibly be reduced specifically by thoroughly analyzing the tasks at stake and the issues they cover. Valuable contributions to this are especially owed to Elmar Cohors-Fresenborg, Johann Sjuts and Lisa Hefendehl-Hebeker. In the end, we were not yet able to develop and implement a sustainable solution. However, we are convinced that it makes sense to strengthen children's functional-logical thinking more explicitly (for more information see the section 'Future Perspectives'). Arithmetic is the core component of elementary school mathematics lessons, so it is not surprising that many of the ZMO tasks, including word problems, require arithmetic thinking. To understand and to cope with arithmetic relations, a good orientation in the number space is necessary, which is based on the arithmetic relations between numbers. In basic mathematical research and philosophy, it has been established that the basis of natural numbers is the formation of successors, and thus a central aspect of numbers is their construction history (see e.g. Cassirer, 1910, Dedekind, 1969/1887, Frege, 1977/1884, Gowers, 2002, Natorp, 1910, Meschkowski, 1969). In addition, an understanding of the written decimal notation was developed. As yet, there are hardly any math learning tools that can be used to make a seamless, direct transition from the action with these to the formal decimal notation, so that arithmetic thinking is particularly required (but see Gallin 2012, Schwank 2013). The essence of the construction of natural numbers (Dedekind-Peano axioms) in combination with their notation by means of the decimal notation form the basis of arithmetic. In both cases, construction processes play the central role, for a sustainable arithmetic understanding, these processes must be mastered cognitively.

Overall, little mathematical formalism is still available in elementary school, so that structural-content-based thinking based on enactive and iconic forms of representation in the sense of Bruner (1973/1964) is significant. In addition, there is the verbal-conceptual explanation using a representative example in the sense of Hefendehl-Hebeker (2001), from which a propaedeutic for elementary algebra emerges, thus providing a link between arithmetic and algebra. Mathematics lessons in primary school thus serve as an essential basis for the development of advanced mathematical thinking. Also, in order to display traces of an emerging algebraic thinking more noticeably, many of the ZMO-tasks explicitly contain a justification part.

TASK OVERVIEW ACCORDING TO RUBRICS

The tasks used are compiled in Schwank (Language versions: 2014/German, 2016/English, 2017/Indonesian, 2019/Chinese, 2020/Persian). They were inspired by task material from schoolbooks for mathematics lessons in grades 3-5 as well as relevant works, e.g. Mathe für kleine Asse (Maths for Little Geniuses, Käpnick, 2001) or Math Olympiad Contest Problems for

Elementary and Middle Schools (Lenchner, 1997).
The ZMO tasks fall into seven different rubrics. The design is based on the respective framework topic for the creative mathematics contributions. The aim was always to give the children a pleasant start into the Mathematics Olympiad. For this purpose, an introductory task was worked out together with the children, the first tasks – such as number sentences – were rather easy to solve, the two most challenging tasks were well allocated and the last task was an enjoyable ending, such as finding a way through a labyrinth.

Rubric 1: Attentive Calculation

This rubric includes tasks for addition and subtraction, including the handling of errors, arithmetic strategies and calculating with yet unknown numbers. An example is the examination of the given calculation 287+423=600, which was discussed in Schwank (2013). Only few boys stood out since they showed an orientation in the number space by recognizing – instead of simply calculating and judging the presented result on the basis of the self-determined result: 600 is already reached with 200+400, therefore the result cannot be correct.

Another example is shown in Figure 1. This is an introductory task that deals with the smart calculation of the sum of 127 and 398. On average, both girls and boys achieved 3.4 points. The calculation method used here is the method of written addition, which the children learn during the 3rd grade.

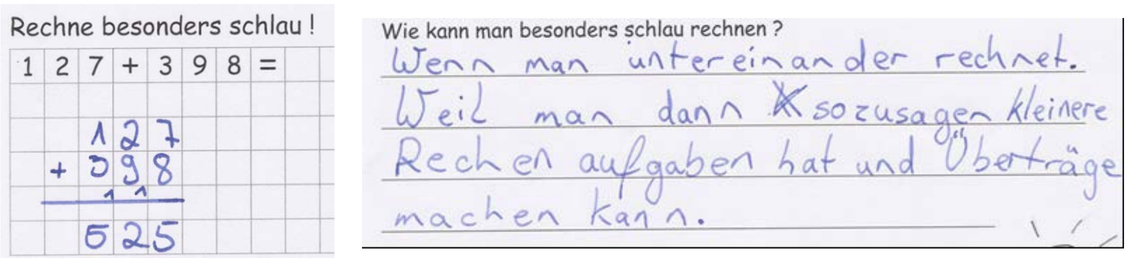


Figure 1. Example of a completion of an introductory task for the Mathematics Olympiad from rubric 1
Left: Calculate especially clever! *Right:* What are especially clever ways of calculating? If you calculate among each other. Because then you have smaller number sentences, so to say, and can make carries.

Rubric 2: Number and Arithmetic Operation Riddles

This rubric includes tasks where missing numbers or arithmetic operations have to be found, as well as number sentences, which are laid with matches and the recognition of number patterns. The task, which natural numbers fulfill the equation □ - ◇ = 2, is discussed in Schwank & Nowinska (2008). How do children succeed in looking at the infinite number of possible number pairs and capturing them with arguments? Again, the way some boys find their way around the number space is noticeable. The children sometimes only name specific examples of numbers, but one boy, for example, writes "Susi could do: 3-1=2, 4-2=2 and up to ▯ - almost ▯ = 2:" or another (in generalization of the task): "The numbers always have to be as much apart as the result". Initiated algebraic thinking is obvious.

Figure 2 shows the completion of a task that was decisive for one of the top places. The three first-placed girls achieved a total of 3 points (1.5; 1.5; 0), the three first-placed boys 16 points (5; 5.5; 5.5). All in all, the average score of all 71 girls was 0.27, that of all 72 boys 0.73. The missing points at the maximum score of 6 points are due to the fact that 6 boys (none of the girls) were able to complete the task completely correctly, but none of them formulated a complete explanation for it.

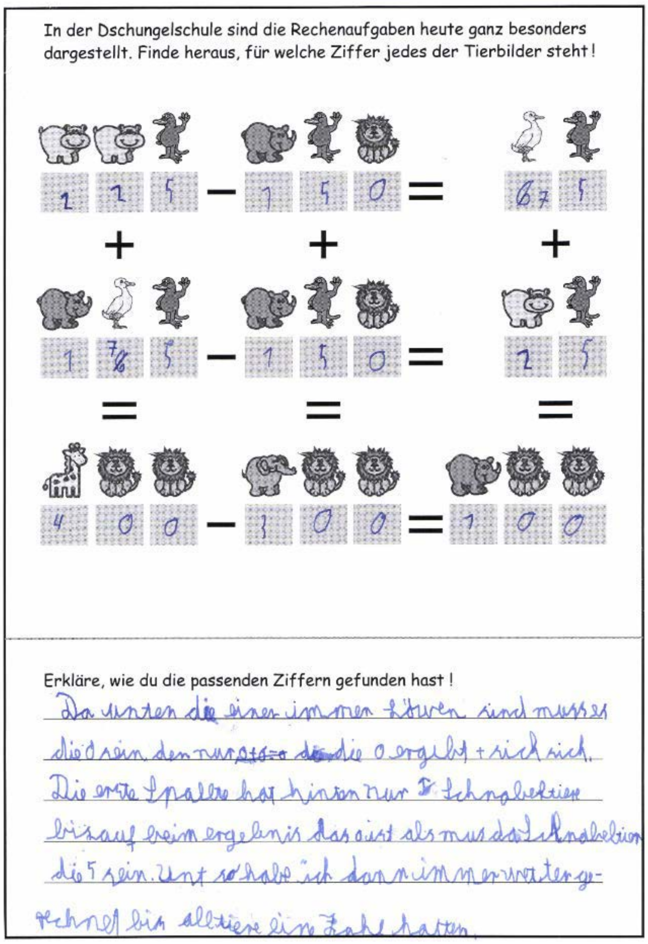


Figure 2. Example of a successful completion of a challenging task from rubric 2. The guiding idea is the reachability of numbers.

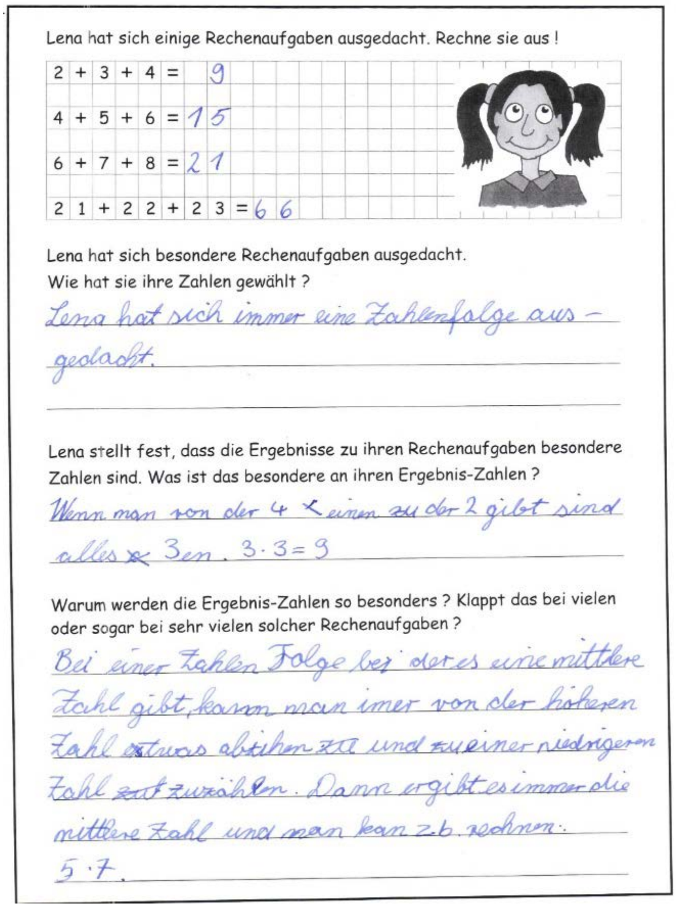
Task: In Jungle School, the number sentences are presented in a very special manner. Find out which digit is represented by each of the animal pictures.

Solution: Since the ones down there are always lions, it has to be the 0, because just ~~0+0=0 the~~ the 0 + itself adds up to itself. The first column has just platypuses in the back, except from the result which is 0, so the platypus has to be the 5. And so, I continued calculating until all animals had a number.

Rubric 3: Arithmetic Regularities

This rubric includes tasks for continuing numerical series as well as addition and multiplication tasks with special properties. In a sequence of the continuation of the series 4, 6, 9, 13, 18, ... 4, 6, 10, 18, 34, ... 3, 6, 5, 10, 9, ... the girls scored on average half a point less than the boys (among the 105 girls: 10.48% full score, 4.76% zero points; among the 115 boys: 14.78% full score, 1.74% zero points).

Another example task is shown in Figure 3, where the girls scored an average of 2.68 points and the boys 2.90 points. Here, the number sentences have to be put into relation to each other. It would be possible to recognize that the results belong to the 3 times multiplication table. The question is whether this always applies in the case of the given common structure of the number sentences or also how the additive calculation can be reinterpreted to a different calculation. In the illustrated completion of the task the boy concentrates on the procedure of the calculation and gives as example that instead of calculating 2+3+4 also the calculation 3×3 is possible, namely by giving one from the 4 to the 2. He still lacks the expression for taking this specific task as a representative example of the general validity of the context. Therefore, he tries to generalize the formulation of the task in the last part of the task: If such a number sequence has a mean number, then the numbers can all be balanced to this mean number and a multiplication task emerges. He does not specify this in more detail, but the statement 5×7 can be interpreted as resulting from 5+6+**7**+8+9 or 2+3+4+**5**+6+7+8.



Lena hat sich einige Rechenaufgaben ausgedacht. Rechne sie aus !

2 + 3 + 4 = 9

4 + 5 + 6 = 15

6 + 7 + 8 = 21

21 + 22 + 23 = 66

Lena hat sich besondere Rechenaufgaben ausgedacht.
Wie hat sie ihre Zahlen gewählt ?

Lena hat sich immer eine Zahlenfolge aus-gedacht.

Lena stellt fest, dass die Ergebnisse zu ihren Rechenaufgaben besondere Zahlen sind. Was ist das besondere an ihren Ergebnis-Zahlen ?

Wenn man von der 4 einen zu der 2 gibt sind alles 3en. 3·3=9

Warum werden die Ergebnis-Zahlen so besonders ? Klappt das bei vielen oder sogar bei sehr vielen solcher Rechenaufgaben ?

Bei einer Zahlen Folge bei der es eine mittlere Zahl gibt, kann man imer von der höheren Zahl etwas abziehen und zu einer niedrigeren Zahl zuzählen. Dann ergibt es immer die mittlere Zahl und man kan z.b. rechnen. 5·7.

Figure 3. Example of a successful completion of a task from rubric 3.
Problem section 1: Lena thought up some number sentences. Solve them!
Problem section 2: Lena thought up very special number sentences. How did she choose her numbers?
Lena has always thought up a numerical series.
Problem section 3: Lena notices that the results of her number sentences are very special numbers. What is special about her result numbers?
If you give one from the 4 to the 2, they are all 3s. 3×3=9
Problem section 4: Why do the result numbers become so special? Does this work with many or even a lot of these number sentences?
In a numerical series, in which there is a mean number, you can always take something away from the larger number and add it to a smaller number. Then it always adds up to the mean number and you can calculate, for example: 5×7.

Rubric 4: Operating with Options

This rubric covers combinatorial tasks, so one has to find appropriate arrangements, search for different possibilities, or even find out all possibilities.

Figure 4 shows an example of the completion of a task. The iconic design of the objects to be put into a combinatorial context is noticeable. The text of the task explicitly indicates the characteristic difference (red, blue, yellow) only for the pants, for the sweaters and pointed caps it is designed imaginatively, e.g. for the sweaters: red cross-striped, blue dotted, green wavy, yellow diagonally striped. With an average of 1.82 points, the girls scored slightly better than the boys, whose average is 1.73 points (among them, from 88 girls 0% full score, 11.36% zero points, from the 94 boys 2.13% full score, 17.02% zero points). Losses of points resulted from wrong approaches or comparatively insufficient justification or clarity of presentation.

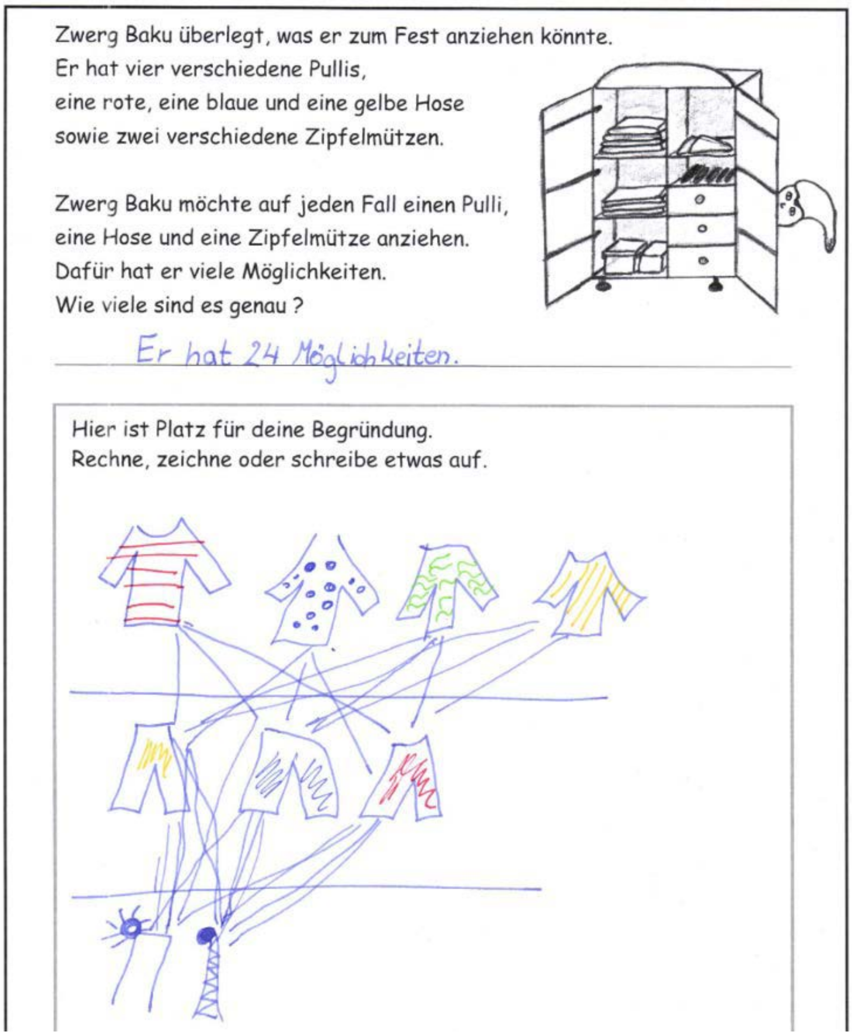


Figure 4. Example of a successful completion of a task from rubric 4

Problem section 1: Dwarf Baku wonders what he can wear to the party. He has four different sweaters, one pair of red, one pair of blue and one pair of yellow trousers as well as two different pointed hats. Dwarf Baku definitely wants to wear one sweater, one pair of trousers and one pointed hat. For this he has many options. How many are there exactly?

He has 24 options.

Problem section 2: Some space for your explanation. Calculate, draw or write something down.

Rubric 5: Word Problems

This rubric covers tasks, where one, two or even more unknown quantities have to be found. It deals with areas, paths and distances, further about the overview of temporally interwoven relations, including situations with the characteristic 'the more, the more' or 'the more, the even more' and finally about multiplicative relations and exploration of if-then-relations.

Figure 5 shows the completion of one of these tasks of which the completion was decisive for reaching one of the top positions. It is remarkable that the backward calculation to determine the number of birds is successful and the calculation is controlled, both for individual calculation steps (results are corrected) as well as with regard to the overall concept (forward calculation as a check). None of the 70 girls were able to solve this task, therefore all receiving zero points. The 72 boys received an average of 0.31 points (including 4.2% almost full points, 93% zero points).

Auf einem großen Urwaldbaum sitzen viele Vögel mit ihrem Anführer Logi. Auf einem Nachbarbaum sitzen die beiden Schimpansen Pa und Pu. Pa ruft zu den Vögeln: "Hallo ihr 200 Vögel!" Logi antwortet: "So viele sind wir nicht. Aber wenn du zu unserer Anzahl das Doppelte hinzufügst und euch beiden mitzählst, dann wären wir 200 auf diesem Baum."

Wie viele Vögel sitzen auf dem Baum ? 66

Hier ist Platz für Deine Überlegungen.
Rechne, zeichne oder schreibe etwas auf.

200-2=198   198:3=66

~~10~~

~~490:3=~~

66·2=132

132+66=198

198+2=200

Figure 5. Example of a successful completion of a challenging task from rubric 5.
Problem section 1: Upon a large jungle tree, many birds are sitting together with their leader Logi. On a neighboring tree the two chimpanzees Pa and Pu are sitting. Pa calls out to the birds: "Hello you 200 birds!" Logi answers: "We are not that many. But if you add to us the double of our number and also count the two of you, then we would be 200 on this tree."
How many birds are sitting on the tree?
Problem section 2: Some space for your thoughts. Calculate, draw or write something down.

Rubric 6: Figurative Patterns

This rubric deals with the recognition and continuation of patterns, the examination of patterns, the number of squares and rectangles, area determination, scaling up, spatial imagining and finally figures to be cut out of a folded piece of paper.

Figure 6 shows a task in which the children could be active, a figure is to be cut out. To loosen up, it was usual to provide such an activity task during the ZMO (except cutting e.g. placing matches or coins). The folded sheet was part of the task papers. The children were asked to bring scissors with them and there were enough spare scissors available. The girls achieved an average of 2.25 points, the boys 2.20 points. In fact, it is not easy to cut out an object that comes close to a circle. In the given case it becomes clear that the child has the imagination to see an unfolded crescent in the circle to be created.

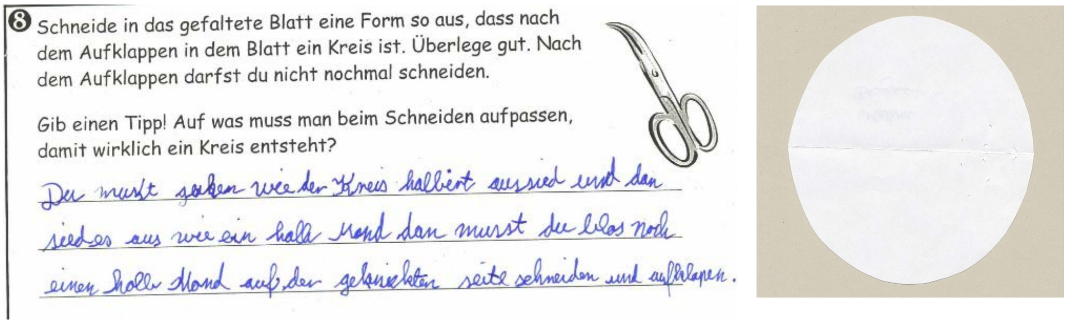


Figure 6. Example of a successful completion of a task from rubric 6.
Task: Take a folded piece of paper. Now cut out a figure in such a way that you see a circle when unfolding the paper. Think carefully. You may not cut again after unfolding.
Give a hint! What is important while cutting in order to get a circle as a result?
Solution: You have to see how the circle looks like halved, and then it looks like a crescent and then you just have to cut a crescent on the folded side and unfold it.

Rubric 7: All's well that ends well

Taking part in an Olympics is exhausting. Just as the children should be well introduced to the ZMO with the introductory task, they should also be well released with a relaxing task, typically a labyrinth task similar to the one shown in Figure 7. Task completion also involves dealing with the task. Interesting here are the children's means of expression. In the present case, the child uses iconic representations in addition to answer sentences to make it clear how he or she imagines making the given labyrinth more difficult or easier.

Figure 7. Example of a successful completion of a task from rubric 7.
Problem section 1: Find the path to the treasure!
The blue path is right.
Problem section 2: How can one design the maze easier or more complicated?
it would be more difficult if you had to drive around ~~first~~.
it would be easier if you could drive straight ahead, then turn right.

FUTURE PERSPECTIVES – JOURNEY INTO THE FUTURE

Mathematical thinking and problem solving is a highly complex terrain that has not yet been understood completely from a cognitive perspective. In particular, it is unclear what exactly could be the causes of differences in problem solving between girls and boys or also which conditions promote successful problem solving. Orientation is provided by publications on gender-specific differences, many of which are now available – for a first overview see e.g. Becker et al. (2007). In this section we want to investigate an important aspect – this in accordance with the above explanations of the meaning of construction processes as a starting point for the foundation of natural numbers and thus of arithmetic. For this we consider a QuaDiPF-task for deductive logical thinking, whose format goes back to Spearman (1904). We have to consider which figure that is missing in the lower right-hand corner can suitably complete a 3x3 arrangement of the so far eight figures. For an example see Figure 8 (Schwank, 1998, 2003).

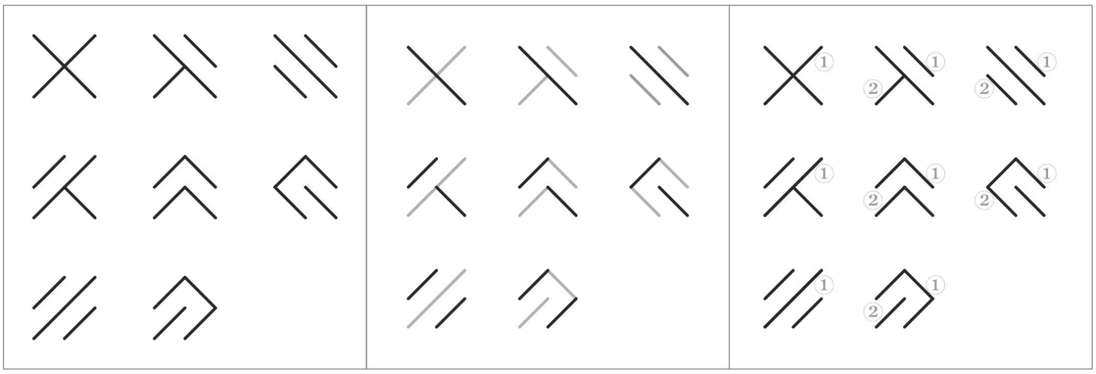


Figure 8. Left: Example of a QuaDiPF task.
Middle: predicative-logical analysis. Right: functional-logical analysis.

It is remarkable that a well-fitting solution figure can be determined using two very different approaches. In a *predicative-logical analysis*, the given figure arrangement is given structure by recognizing common elements, which can then be used to assume a solution figure: In this case the 'diagonals' on one hand remain the same line by line (black lines) and on the other hand column by column (grey lines). Thus, the solution figure results as a composition of the 'diagonals' to be considered as equal in the last row and column. In a *functional-logical analysis*, the elements connecting the figures are actions, which cause the figures to emerge from each other in a temporal sequence: Line by line, first the upper right line is rotated, then the lower left line. Column by column, this happens first with the upper left line, then with the lower right line. The solution figure results from applying this change process to the original figure in the last row or column. In the predicative-logical case, the core of cognitive reasoning is the engagement with the static structural elements, in the functional-logical case, the engagement with the dynamic construction processes. The ability to functionally and logically manage conditions is obviously useful to find one's way sensibly into the number space given by the natural numbers: By suitable constructions you can get from given numbers to others, by means of constructions relations between numbers can be made. From the beginning it became apparent that boys tend more to functional-logical analysis than girls (Schwank, 1994).

This finding harmonizes with known research results and places functional- or predicative-logical thinking in the context of other constructs examined. Three such results are discussed here:

(1) *Mental rotation*: In the original version, two-dimensional drawings of three-dimensional cube chains are to be examined to determine whether they are different or the same cube chains that are merely rotated differently in space or (additionally) mirrored (Shepard & Metzler, 1971). This task format has become famous not only because it provides evidence for cognitive language-independent performance, but also because numerous studies have shown highly significant differences in performance in favor of men

compared to women (e.g., Voyer, Voyer, & Bryden, 1995; on more recent aspects with variations in task presentation see e.g., Fisher, Meredith & Gray, 2018). A limited capacity for mental rotation makes the functional-logical analysis of a QuaDiPF-task as shown in Figure 8 difficult. This is an example of the decomposition of the ability for functional-logical thinking into its sub-components or basic cognitive skills necessary for it.

(2) *Different playing behavior*: The development of cognitive abilities is (besides e.g. genetic predispositions) experience dependent. Research indicates that the cognitive development of boys, at least in one relevant area, playing behavior, is bound to different experiences than that of girls starting from an early age (see e.g. Ruble, Martin & Berenbaum, 2006): This is not only expressed in significant findings regarding the preferred choice of toys that are used: e.g. transport vehicles, balls in the case of boys; dolls, puzzles in the case of girls – but also in play practice: e.g., games with increased physical activities (such as romping, scuffling, climbing) in the case of boys; physical activities restricted to a smaller space (such as dollhouses, dolls' kitchen games) in the case of girls, with a comparatively increased verbal component. It remains to be further investigated how certain activity experiences affect cognitive abilities. With regard to the development of functional-logical thinking, a special focus should be placed here.

(3) *Language skills*: There are numerous studies that show a difference in favor of women (see e.g. Ullman, Miranda & Travers, 2007). Among other things, it is assumed that women benefit from declarative memory, while men benefit from procedural memory. Regarding the QuaDiPF tasks it is noticeable that functional-logical analyses often use deictic language without a closer linguistic grasp of the objects being treated (that happens with this one) while predictive-logical analyses use words like containers in which similar objects are collected (set formation, use of terms like: these diagonals). The constructor of the above example QuaDiPF task (Fig. 8), for which the lines of the initial figure simply fold over to form a flat figure, was surprised to note that at first glance, Johann Sjuts identified the invariance of the diagonals of the individual figures as an element manifesting structure.

Dealing with gender differences is a delicate matter. Nevertheless, we do see a need to pay more attention to the diversity of children in general. In the future, the challenge will be to explore the individual cognitive foundations of mathematical thinking and problem solving more intensively and thus to work towards making the address-oriented, instructional development of these skills even more successful.

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REFERENCES

Becker, J. B., Berkley, K. J., Geary, N., Hampson, E., Herman, J. P., & Young, E. (Eds.) (2007). *Sex Differences in the Brain: From Genes to Behavior*. Oxford: Oxford University Press.

Brainerd, C. (1979). *The Origins of the Number Concept*. New York: Praeger.

Bruner, J. (1973/1964). Der Verlauf der kognitiven Entwicklung. In D. Spanhel (Hrsg.), *Schülersprache und Lernprozesse*. Düsseldorf: Schwann.

Cassirer, E. (1910): *Substanzbegriff und Funktionsbegriff. Untersuchungen über die Grundlagen der Erkenntniskritik*. Berlin: Verlag von Bruno Cassirer.

Dantzig, T. (1930). *Number, the Language of Science*. New York: The Macmillan Company.

Dedekind, R. (1969/1887). *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg.

Fisher, M.L., Meredith, T. & Gray, M. (2018). Sex Differences in Mental Rotation Ability Are a Consequence of Procedure and Artificiality of Stimuli. *Evolutionary Psychological Science*, 4, 124–133.

Frege, G. (1977/1884). *Die Grundlagen der Arithmetik: eine logisch-mathematische Untersuchung über den Begriff der Zahl*. 2. Nachdruckauflage. Hildesheim: Georg Olms.

Gallin, P. (2012). *Die Praxis des dialogischen Lernens in der Grundschule. Handreichungen des Programms SINUS an Grundschulen*. Kiel: IPN.

Gowers, T. (2002). *Mathematics. A very short introduction*. Oxford: University Press.

Hefendehl-Hebeker, L. (2001). Die Wissensform des Formelwissens. In W. Weiser, B. Wollring, B. (Hrsg.): *Beiträge zur Didaktik der Mathematik für die Primarstufe. Festschrift für Siegbert Schmidt*. (S. 83-98). Hamburg: Verlag Dr. Kovac.

Lenchner, G. (1997). *Math Olympiad Contest Problems for Elementary and Middle Schools.* East Meadow, NY: Glenwood Publications.

Meschkowski, H. (1969). *Wandlungen des mathematischen Denkens. Eine Einführung in die Grundlagenprobleme der Mathematik*. Braunschweig: Friedr. Vieweg & Sohn.

Natorp, P. (1910). Die logischen Grundlagen der exakten Wissenschaften. Leipzig: Teubner.

Käpnick, F. (2001). *Mathe für kleine Asse*. (Handbuch für die Förderung mathematisch interessierter und begabter Dritt- und Viertklässler). Berlin: Volk und Wissen.

Ruble, D. N., Martin, C. L., & Berenbaum, S. A. (2006). Gender Development. In N. Eisenberg (Ed.), *Handbook of Child Psychology. Volume 3. Social, Emotional, and Personality, Development* (6th ed. pp. 858-932). New York: Wiley.

Schwank, I. (1994). Zur Analyse kognitiver Mechanismen mathematischer Begriffsbildung unter geschlechtsspezifischem Aspekt. ZDM-Analysenheft "Frauen und Mathematik". *Zentralblatt für Didaktik der Mathematik*, 2, 31-40.

Schwank, I. (1998). *QuaDiPF. Qualitatives Diagnoseinstrument für prädikatives versus funktionales Denken.* Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Schwank, I. (2003). Einführung in funktionales und prädikatives Denken. *Zentralblatt für Didaktik der Mathematik*, 35(3), 70-78.

Schwank, I. (2013). Kleine Einsen und ein Wunderwerk. Die Zwergen-Mathe-Olympiade. *Grundschule,* 11, 16-19 & IX-XV (Beihefter „Grundschule Extra").

Schwank, I. (2014). *Kinder in ihrem mathematischen Talent wertschätzen – Olympische Aufgabensammlung*. Osnabrück: Forschungsinstitut für Mathematikdidaktik.
2018: 2. überarbeitete und erweiterte Auflage.
Further language versions:

- Schwank, I. (2016). *Cherishing Children in their Mathematical Talent – Collection of Olympic Tasks.* Osnabrück: Forschungsinstitut für Mathematikdidaktik.
- Schwank, I. (2017). *Menghargai anak-anak dalam bakat matematika mereka – Koleksi Tugas Olimpiade.* Osnabrück: Forschungsinstitut für Mathematikdidaktik.
- 英格・施万克（2019）．珍惜孩子的数学天赋 – 奥林匹克数学习题汇编．奥斯纳布吕克：数学教学理论研究中心.
- شوانک، ای. (۲۰۲۰). پرورش استعدادهای ریاضی در کودکان - مجموعه‌ای از تکالیف المپیک. اسنابروک: موسسه تحقیقاتی آموزش ریاضی

Schwank, I., & Nowinska, E. (2008). Die Denkform des Formeldenkens. In B. Barzel, T. Berlin & A. Fischer (Hrsg.): *Algebraisches Denken. Festschrift für Lisa Hefendehl-Hebeker* (S. 111-122.) Hildesheim: Franzbecker.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense – making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (S. 334-370). New York: MacMillan.

Shepard, R., & Metzler, J. (1971). Mental rotation of three-dimensional objects. *Science*, 171, 701–703.

Spearman, C. (1904). General intelligence, objectively determined and measured. *American Journal of Psychology* 15, 201-293.

Ullman, M. T., Miranda, R. A., & Travers, M. L. (2007). Sex Differences in the Neurocognition of Language. In J. B. Becker, K. J. Berkley, N. Geary, E. Hampson, J. P. Herman & E. Young (Eds.) (2007). *Sex Differences in the Brain: From Genes to Behavior* (S. 291-311). Oxford: Oxford University Press.

Voyer, D., Voyer, S., & Bryden, M. P. (1995). Magnitude of sex differences in spatial abilities: a meta-analysis and consideration of critical variables. *Psychological Bulletin*, 117, 250–270.