European Research in Mathematics Education I.II

Inge Schwank (Editor)

Forschungsinstitut für Mathematikdidaktik, Osnabrück 1999
ISBN: 3-925386-51-3

Die Deutsche Bibliothek - CIP-Einheitsaufnahme
Schwank, Inge: European Research in Mathematics Education I.II
- Proceedings of the First Conference of the European Society in Mathematics Education Vol. II; Internet-Version
ISBN: 3-925386-51-3

Layout (including cover): By means of Corel Ventura ®

Please note:
Printed versions will be available:
Libri Books on Demand
Vol. I: ISBN: 3-925386-53-X
Vol. II: ISBN: 3-925386-54-8
Vol. III: ISBN: 3-925386-55-6

Internet-Versions:
Vol. I: ISBN: 3-925386-50-5
Vol. II: ISBN: 3-925386-51-3
Vol. III: ISBN: 3-925386-52-1

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MATHEMATICAL THINKING AND LEARNING AS COGNITIVE PROCESSES

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The work was prepared in such a way that - after the reviewing process - all accepted papers were distributed to the prospected group members. In the spirit of CERME the group leaders decided that during the sessions the accepted papers would not be orally presented one by one. For the purpose of the stimulation of a goal-orientated, in depth discussion six general themes had been identified and two members of the group were asked to give a general introduction referring to the papers fitting each theme and to current research developments. The themes and the introduction presenters are as follows:

1. *The Nature of Cognitive Structures - Introduction and Overview*
   Emanuila Gelfman & Inge Schwank
2. *Individual Styles of Cognition*
   Sara Hershkovitz & Marina Kholodnaya
3. *Cognition and Emotion/Motivation*
   Elena Nardi & Rosetta Zan
4. *Cognition and Language*
   Pierre Luigi Ferrari & Pearla Nesher
5. *Cognition and New Technologies*
   Tatiana Oleinik & Elmar Cohors-Fresenborg
6. *Mathematical Thinking in Modern Conditions – Situated Cognition*
   Jarmila Novotna & Elena Nardi

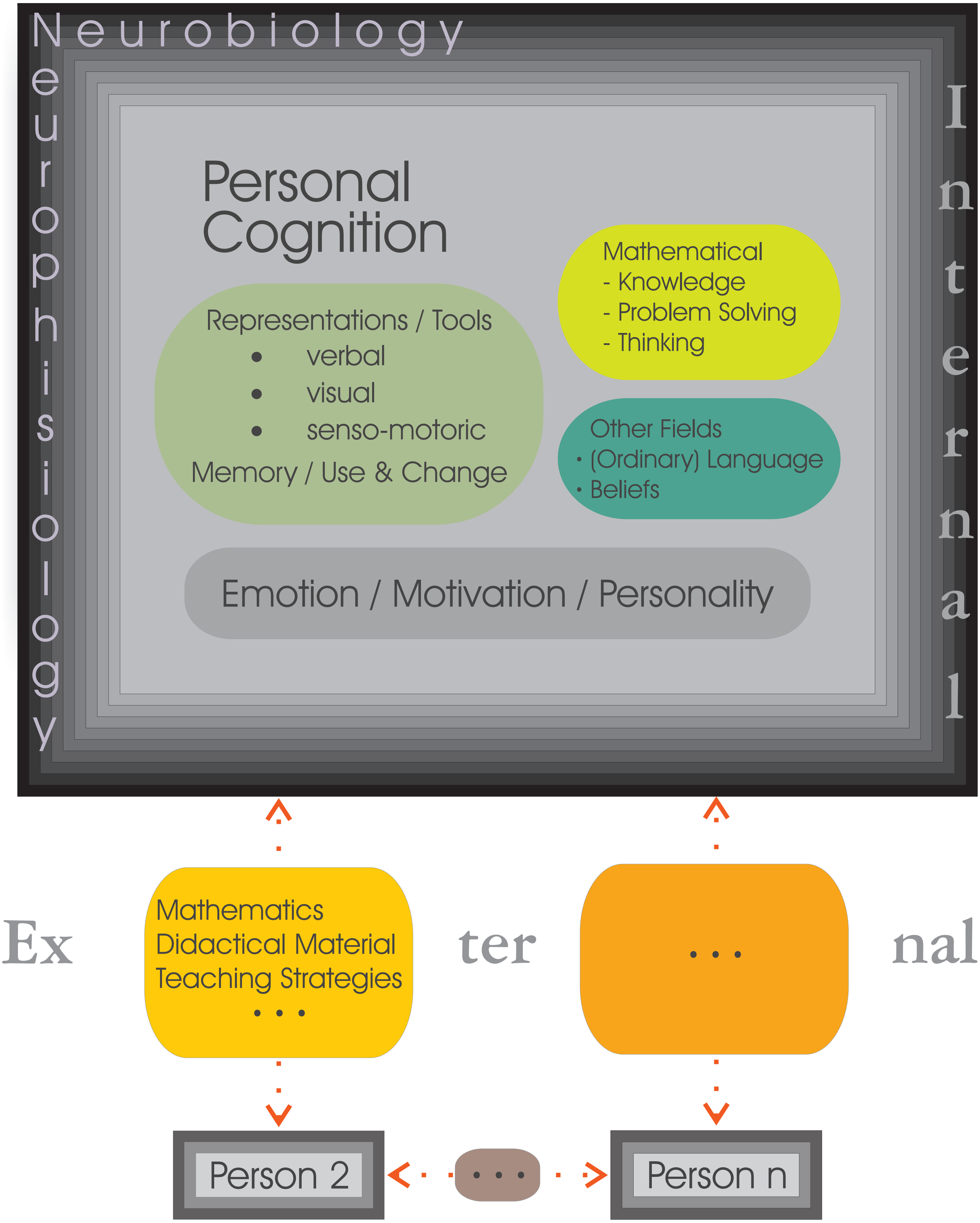

Fig. 1: The Nature of Cognitive Structures - Personal Cognition

1. The Nature of Cognitive Structures - Introduction and Overview

1.1 General framework

At the beginning of the group work Inge Schwank gave an introduction and developed a general framework (Fig. 1) in which personal cognition could be considered under different aspects.

One aspect was the role which different kinds of representation of concepts like visual, verbal and sensor motoric play in the process of concept formation and how these are correlated to mental tools. A second aspect dealt with the field of different aspects of mathematical learning like the matter of mathematical knowledge, the specific mental actions in problem solving and the nature of mathematical thinking. It was worked out that these specific mathematical aspects had to be considered in their correlation to other fields, especially cognition and the use of ordinary language. These different aspects of one person's cognition are controlled by the person's beliefs. This is the bridge between cognition and the role of emotion, motivation and the person's personality. Inge Schwank mentioned that these psychological reflections had to be based on some outcomes of neurophysiology and, even more, on neurobiology. All these aspects discuss cognition under the internal perspective of a person. Of course, in studying mathematical teaching and learning processes this point of view has to be broadened to an external perspective in which different aspects of mathematics, didactical materials or teacher strategies are some examples which influence one person's cognitive processes. A broader perspective comes into consideration if one studies interactions of persons, either among learners or between teachers and learners.

1.2 Individual Preferences - Predicative versus Functional Cognitive Structures

In the centre of discussion there was Inge Schwank's theory concerning the distinction of predicative versus functional cognitive structures and the hypothesis that most people have a more or less strong preference for one of these two cognitive structures in which they model, understand and solve given external problems. Inge Schwank pointed out that this distinction had its roots in different aspects of mathematics. As one example she explained how famous mathematicians stress the aspects that mathematical concepts have to be understood through the glasses of process orientation and that concepts very often play the role of tools and not of statements. Relations to some of the closest theoretical concepts were discussed, e.g. declarative - procedural knowledge, procepts, APO-Schema (action→process→object - rule).

1.3 Cognitive Experiences - Information Coding

Emanuila Gelfman treated composition of some forms of students' mental experience: cognitive, metacognitive and intentional. Students' cognitive experience is paid special attention. In particular, Emanuila Gelfman gave the examples of how, using specially constructed school-texts for 10-15 year old students, we may introduce different forms of information coding: verbal, visual, sensual-sensory. Then she pointed out the role which different ways of information coding plays in students' intellectual developments and dealt with the advantages and difficulties which arise from verbal versus formal representation of mathematical concepts and problems.

2. Individual Styles of Cognition

This aspect of cognition was tackled under two complementary points of view. Sara Hershkovitz started her presentation from examples of analysing pupils' behaviour when they are dealing with specific word problems.

In a more general talk Marina Kholodnaya explained - from a psychological point of view - which different aspects of individual styles had to be considered: information coding styles, information processing styles, solving problem styles, epistemological styles. There was a fruitful discussion concerning the question how the theory of Inge Schwank concerning individual preferences for cognitive structures fitted into the general psychological discussion concerning individual styles of cognition.

3. Cognition and Emotion / Motivation

In general Elena Nardi and Rosetta Zan stressed the importance which emotion plays in controlling the process of cognition. They pointed out that failure in solving problems is not only due to a lack of knowledge, but also to the incorrect use of knowledge which is often inhibited by both, general and specific beliefs: about mathematics, about self, about mathematics teaching, about social context.

Elena Nardi exemplified research and literature in the field of cognition and emotion and emphasised the inextricability of emotional and cognitive factors in the formation of conceptual structures. Current literature suggests links between attitudes towards mathematics as a lifetime utility tool and the development of conceptual understanding, between problem solving skills and beliefs about the nature of a mathematical problem, between development of formal mathematical conceptual structures and beliefs about the necessity of mathematical proof etc.

Rosetta Zan refined the discussion of affective factors by introducing the distinction and interplay between beliefs, attitudes and emotions. She revisited the submitted papers by pointing out how the cognitive analysis presented in them was substantially incorporating a consideration of beliefs, attitudes and emotions. She subsequently presented a vista of international works in the area and concluded with potential future research questions in the area.

4. Cognition and Language

Pearla Nesher gave an overview concerning research in the solving of word problems under the perspective of language. She pointed out that different wordings may induce different mental models of the given problem in the pupil’s mind. The study of these interferences gives fruitful hints for the explanation of pupils misconceptions and difficulties in mathematics lessons.

Pierre Luigi Ferrari analysed typical mistakes and difficulties of senior students while using mathematics language. Pearla Nesher and Pierre Luigi Ferrari put the question of necessity of special work of students for mastering mathematics language.

5. Cognition and New Technologies

Although there had been a thematic group concerning new technologies the group leaders had decided that in this thematic group concerning cognition there should be one discussion concerning the aspect which role the access to new technologies plays for the development of mathematical cognition and the ways of teaching and learning mathematics. Tatiana Oleinik pointed out how in the practical teaching process specific software like Derive and Cabri geomètre foster students' cognitive growth. She explained how the use of these software packages can support visual thinking as an important aspect of mathematical thinking and how it can support an open end approach in mathematics teaching.

Elmar Cohors-Fresenborg tackled the problem of the session from a more fundamental point of view. He discussed the interaction between external presentations and the use of mental models under the perspective that the use of computers enables a more visual or process-orientated external representation. He pointed out that according to the distinction between predicative versus functional cognitive structures the support of the dynamic aspect of using computers may support the more functional-orientated students while the traditional way of using static formalisations on paper had supported the predicative ones.

In the following discussion there was a strong debate concerning the question how far the use of computer in the mentioned way - really from a cognitive point of view - has to be distinguished from the traditional way of dealing with mathematical representations on paper. It was pointed out that the specificity of a functional mental model cannot be represented in a static representation on paper, e.g. neither the concept of balancing of weight nor the process of switching and its consequence in a circuit.

6. Mathematical Thinking in Modern Conditions – Situated Cognition

In her introduction Elena Nardi pointed out the increasing international interest in Situated Cognition. Jarmila Novotna then opened up four areas of discussion related to the use of relevant projects in the mathematics classroom: time considerations, a definition of a "project", projects as social problems and the issue of terminology linked with projects.

Subsequently the group debated on the links between situated cognition and the development of formal mathematical reasoning. Possible risks were highlighted with regard to the excessive embedding of the learner's mathematical reasoning in the situational structure of a project and alternatives were also brought into consideration.

7. Benefit and Outcome of the Working Group

The discussions in the official group sessions induced in several cases the wish for a more private and more intensive discussion with the target to understand the colleagues' theoretical positions and to go deeper into the different experimental designs.

The discussion showed that there could be different ways of co-operation. One deals with the question how far behind different wording of theoretical explanations there are common aspects. A further intensive discussion was to make clear where the similarities and distinctions between the different theoretical framework lie.

The second possibility of co-operation deals with the idea that an observed behaviour of mathematical activities can be explained by different scientists using their different theoretical explanations: Two pairs of scientific eyes should provide a better picture of the phenomenon which leads to a deeper understanding.

The third aspect for future co-operation concerns the question how to replicate an experimental design in different countries. This experimental approach is common in science, but should be more often used in mathematics education.

Another aspect of further co-operation could be the possibility of adding a specific situation in an experiment to an additional experimental design from a colleague of a foreign country. One benefit of such extension could be that both partners have a better insight into the cultural variances and invariances and the importance of different curricula or school systems for the analysis and understanding of the mathematical teaching and learning processes.

COOPERATIVE PRINCIPLES AND LINGUISTIC OBSTACLES IN ADVANCED MATHEMATICS LEARNING

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Abstract: *This paper is concerned with the role of language in advanced mathematical thinking. It is argued that some behaviors may arise from the application to mathematical language of some conventions of ordinary language. Grice's Cooperative Principle (CP) is introduced in order to explain some episodes that are not easily accounted for in terms of semantics only. Some examples of (undue) application of CP to mathematical language are given. It is argued that the application of CP to mathematical language in problem solving is closely linked to the poor use of mathematical knowledge and, more generally, to the attitudes and behaviors that Vinner (1997) names 'pseudo-analytical'.*

Keywords: *language, problem-solving, pragmatics.*

1. Introduction

The opinion that failure in mathematics is often connected to language is increasingly popular among mathematics educators and researchers in mathematics education. From the one hand it is almost common to find some correlation between mathematical performance and general linguistic competence, from the other hand a number of studies have pointed out specific obstacles related to mathematical language, in particular to the interpretation of mathematical expressions and the translation of ordinary statements into mathematical formalism.

A fair amount of research has been devoted to the analysis of syntax, semantics or vocabulary of algebraic language in comparison with natural language. For example, a number of studies have dealt with the so-called *reversal error*, regarded as an example of improper translation. In particular, Bloedy-Vinner (1996) ascribes to the lack of predicates in mathematical language some improper translations into equations of the relationships described in the statements of word problems. In another line of research, Duval (1995) shows that the lack of congruence between representations strongly affects translations between natural and symbolic language. These streams of research have no doubt provided convincing interpretations of students' behaviors and new teaching ideas. Nevertheless there are behaviors which cannot be accounted for this way. Let us see two examples. Problems 0.1 and 0.2 have been administered to samples of freshman computer science students from 1994 to 1997.

Problem 0.1: Is it true that the set A={–1, 0, 1} is a subgroup of (**Z**,+)?

There are students who claim that A is closed under addition and show that if an element of A is added to another (different) element of A, the result belongs to A. They do not take into account `x=y=1` nor `x=y=–1`, the only evaluations which lead to discover that sum is not a function from A×A to A. In other words, they misinterpret the definition of subgroup. This happens in spite of students' knowledge of different representations (addition tables, …) which point out that an element can be added to itself. Moreover, they seem aware that each of the variables `x`, `y` may assume any value in A. Most likely, they are hindered by the need of using two different variables to denote the same number, which does not comply with the conventions of ordinary language, according to which different expressions (in particular, atomic ones) usually denote different things.

Problem 0.2: Let m, n be integers such that m·n = 4·6; can you conclude that m=4 and n=6?

Affirmative answers are almost frequent. Most of students, if interviewed after the test, would explain their answer with arguments like *"You wrote 4·6 instead of 24; there must be some reason for this."*

Behaviors like these can hardly be ascribed to the wrong interpretation of specific expressions or symbols, nor they can be viewed as translation errors; they are often classified, by university and high school teachers as examples of carelessness or naïveté, but a closer analysis shows that they can be regarded as the result of improper application of conventions and forms of ordinary language (including the less accurate ones) to mathematics. In both examples students' behaviors are linked to the interpretation not only of expressions and statements, but also of the interaction as a whole and the goals of the interlocutor.

Some ideas and results of pragmatics may help to interpret behaviors like these. I do not expect that pragmatics can provide a definite theory explaining phenomena that are specific of mathematical language and theories. What I search for are new perspectives in order to interpret some behaviors and to start an alternative analysis of mathematical language. In this paper I try to explain some episodes in terms of Grice's Cooperative Principle (1975), according to which a conversation is not a sequence of incoherent remarks but a cooperative work whose participants recognize a common goal or a mutually shared perspective. The goal may be fixed from the beginning or may develop, but there are some behaviors that are improper from a conversational viewpoint anyway. According to the Cooperative Principle (CP), the contribution of each interlocutor to the conversation is such as required (at the stage it happens) by the shared goals and perspectives of the linguistic exchange. This means that, generally speaking, the amount of information conveyed should be neither more nor less than required by the situation and that, in most situations, only true sentences are to be exchanged: for example, it is common to read or hear (in textbooks, school practice, …) statements like *"You cannot write '2+2=5'"* in place of the more correct *"'2+2=5' is false"*. Moreover, each interlocutor should be (relatively) pertinent, clear, concise, neat, …. When interpreting a statement, one generally supposes that the interlocutor is applying CP. Thus sentences like *"Now Mr. X is not beating his wife"* (which are often true) should violate the principle if Mr. X usually does not beat his wife, and the only interpretation compatible with the presupposition that the interlocutor is applying CP *implicates* that Mr. X usually does that. Mathematical language, for different reasons, often violates CP and students may happen to choose interpretations which are compatible with the presupposition that the interlocutor is applying the principle even if they are mathematically inconsistent or meaningless. So the only interpretation of the use of 2 different variables which is compatible with CP requires that they denote different objects, and the only interpretation of the use of the expression '4·6' in place of the more common and simple '24' requires that it should be related to some purpose of the writer.

I am not claiming that Grice's theory is the most suitable contribution from pragmatics for the analysis of mathematical language (see Caron, 1997, for some critical remarks). Moreover, most theories of natural language understanding are linked to different perspectives within cognitive psychology, which cannot be discussed here. Other frameworks could be used, such as Sperber & Wilson's theory of pertinence (1986) or Clark & Marshall's theory of common ground (1981). But Grice's theory, for its simplicity, seems appropriate to carry out an exploratory analysis of some episodes related to mathematical language from the standpoint of pragmatics. A thorough analysis of mathematical language from the standpoint of pragmatics is yet to be carried out; I think it should start from a keen interpretation of use and functions of mathematical theories (in particular, mathematical logic) and mathematical symbolism.

In sections 1 and 2 some examples of situations will be presented which could be accounted for with the help of CP. Students do not apply CP to interpret any statement or situation. In some cases they seem to interpret expressions according to standard mathematical semantics. Different parts of the same statement may happen to be interpreted in different ways, maybe according to their assumed importance or their exterior aspect. So the parts of the statement that look important may be interpreted according to students' mathematical knowledge, whereas the others are interpreted according to conversational schemes. In section 3 an experiment will be presented which shows that even slight variations of the statement of a problem as well as of punctuation or layout may affect students' resolution procedures as far as students are prevented from choosing conversational interpretations.

2. The Interpretation of Inequality Symbol

In many contexts students are troubled by statements containing the symbols '≤' or '≥'. Dealing with questions like *"Is it true that 2≤100?"* answers like *"No, it is true that 2<100"* are almost frequent. Students' uneasiness may depend on the manifest violation of CP: since 2 is clearly much smaller than 100, why to mention the case 2=100, which is obviously absurd? This may be related to the specific English (and Italian) translation of the symbol '≤', which is usually read as 'less or equal' (Italian:

'minore o uguale'), and then looks like something more complex and expensive (as it requires the writing or the utterance of 3 words in place of 1) than the plain 'less'.

The following problem has been given to a group of 34 computer science majors who had already taken at least 4 mathematics units. The problem has been given within a Mathematical Logic course, after students had been taught some foundations of propositional logic (connectives, truth-table semantics, derivations) and in particular the meaning of words like 'logical consequence', 'incompatible', 'independent' and so on.

Problem 1.1: Let n be a natural number satisfying both of the following conditions:

(1) If $n<100$, then n is even (2) If n is even, then $n>100$

For each of the following statements say whether it is a logical consequence of conditions (1), (2), or it is incompatible or independent.

(a)	$n<100$	*[incompatible]*	(b)	$n\geq 100$	*[consequence]*
(c)	n is even	*[independent]*	(d)	$n>100$	*[consequence]*
(e)	$n\geq 80$	*[consequence]*			

For each of the statements (a), (b), (c), (d), (e) students could choose among 4 pre-arranged answers: *consequence, incompatible, independent, other*.

Here we are more interested in the consistency of the answers than in their global correctness. Let us see the pair (a), (b): answers claiming that one of them is consequence of the conditions and the other one incompatible or that they are both independent have been classified consistent, no matter whether correct or not. All the other answers have been classified inconsistent. Only 17 consistent answers were found. The other 17 students gave inconsistent answers: most of them claimed that (a) and (b) are both consequences of or both contradictory with the conditions (1), (2). Some others claimed that (a) is incompatible (which is correct) and (b) independent (which cannot hold). If we focus on the pair (b), (e), we find 18 inconsistent answers (i.e. those claiming (b) incompatible and (d) independent or consequence, or claiming (b) independent and (d) consequence). If we focus on the pair (b), (e), we find 3 inconsistent answers only.

Behaviors like these may be regarded as consequences of the interpretation of the statements according to conversational schemes. Only the links among (b) and (e) are pointed out even by a superficial interpretation of '≤'. When interviewed, almost all the students involved based their interpretations of '≤' on common language and recognized as obvious the transitivity of '≤'. The same does not work for (b) and (d), whose comparison requires a subtler interpretation (even from a logical viewpoint) of symbols '<' and '≤'. Even the comparison between (a) and (b) requires to recognize that each of them is equivalent to the negation of the other (since **N** is linearly ordered by ≤); this involves some knowledge about **N** which is not explicitly stated in the conditions (1), (2) and is to be expressed. Maybe the relatively large number of students claiming that both (a) and (b) are incompatible focused on the conditions (1), (2) only without using their knowledge about **N**.

3. A Divisibility Problem

Students can be strongly affected by the wording of the problems they are given. In the following example, the task required to answer some questions on an unknown number M, characterized by some conditions, expressed either in Italian language or by formulas, according to the version.

Problem 2.1.: M is a positive integer satisfying all of the three following conditions:

(Version A)

- there exists an integer p such that M=2p+1;
- there exists an integer q such that M=5q;
- there exists an integer k such that M=7k.

(Version B)

- M is odd;
- M is a multiple of 5;
- 7 is a divisor of M.

For each of the following statements determine whether it is true, false or other. Explain.

(c) M+7 is divisible by 14 *[true: M+7 is even and divisible by 7]*

(e) M+7 is a multiple of 21 *[independent: both 35 and 105 fulfill the conditions]*

The pre-arranged answers were: ‘true’, ‘false’, ‘cannot answer, more data required’ and there was a blank space for explanations. The problem has been given to a sample of 35 freshman computer science students; 17 subjects took version A, 18 version B.

	question (c)		question (e)	
	A	B	A	B
Correct with some suitable explanation	10	7	2	9
Correct with no suitable explanation	7	4	0	5
wrong or missing	0	7	15	4
Totals	17	18	17	18

Tab. I: Answers to problem 2.1

Students with version A performed better on question (c), whereas students with version B performed better on question (e). As regards question (c), the most difficult step was seemingly to recognize that M+7 is even. Most of students giving a wrong answer to question (c) recognized that M+7 is divisible by 7 (mainly by means of some equation like M=7k) but failed to recognize that it is even. The availability of equations in version A seemingly helped the students to coordinate divisibility by 2 with divisibility by 7, even if 7 of them were not able to explain their procedure.

As regards question (e) subjects with version A generally did not try to interpret the data and design some strategy, but performed calculations or formal manipulations only; the presentation of data by means of equations might have induced many of them to believe that the answer should be found by means of formal manipulations and that it should be a definite one (‘true’ or ‘false’).

Version B induced the subjects to apply everyday-life reasoning patterns; this version presents some similarity between the external aspect of the conditions and their mathematical meaning: divisibility by 3 is not mentioned nor implied, as it is semantically independent from the conditions. Thus even superficial interpretations of the statement might lead to correct answers. In both cases the subjects have been strongly influenced by the version of the problem.

4. Terms and Objects

All the examples in this section regard the relationship between mathematical expressions and the objects they represent. As already remarked, the use of two different nouns to denote the same object is a violation of CP. This may trouble students when it is necessary to give the same value to two or more distinct variables within an expression. Let us see one more example.

Problem 3.1: Let x, y be divisors of 7; can you conclude that x·y divides 7?

It is not unlikely to find answers like: *"Yes, because 7 is a prime, then the divisors of 7 are 1 and 7, then x=1 e y=7 or x=7 e y=1, in both cases x·y=7."* The case x=y=7 is not taken into account. This problem is even trickier than problem 0.1, because here the correspondence between names (x, y) and things (1, 7) is 1-1 and, if x=y, the statements 'x divides 7' and 'y divides 7' become the same, which makes the violation of CP more evident.

4.1 The Role of Problem Presentation in Problem Solving

Similar results are found in problem solving if we break the correspondence between the clauses which express the conditions to be satisfied and the conditions themselves; this happens, for example, when more clauses express the same condition. Let us consider the following problem.

Problem 3.1A: Find out, if possible, a polynomial with real coefficients, of degree 4, with at least 1 integral root, at least 2 real roots and at least 1 complex, non real root.

An almost frequent answer (by freshman computer-science students) is the following: *"A polynomial like that does not exist, because it should be of fifth degree,* *for it must have 1 integral root, 2 real roots, 2 (conjugate) complex non real roots."* Students do not realize that the first two conditions ('at least an integral root', 'at least 2 real roots') designate 3 roots but actually require 2 only. The relevant aspects of this answer are the following:

- students seem not aware that $\mathbf{Z} \subseteq \mathbf{R}$;
- they seem to focus on the (true) fact that the presence of a complex, non real root implies the existence of another complex non real root (its conjugate).

Explanations based on students' lack of care or distraction are not satisfactory. Actually, if we change some features of the problem we find completely different behaviors. The same students would answer correctly to questions like the following, asked before or after dealing with problem 3.1A.

- Is it true that $\mathbf{Z} \subseteq \mathbf{R}$?
- Are there integers that are not real?
- Is it possible to find an x such that $x \in \mathbf{Z}$ but $x \notin \mathbf{R}$?

Let us consider a problem of the same kind, given to the same groups of students.

Problem 3.2: Find out, if possible, a polynomial with real coefficients, of degree 2, with at least 1 integral root and at least 2 real roots.

Usually, this problem is properly solved by a much larger number of students than 3.1A. The number of students correctly solving problems like 3.1A is always less than 50% of the sample (sometimes even less than 20%), whereas problems like 3.2 are generally solved by more than 80% of the sample.

These data suggest 2 remarks:

- The behavior reported about problem 3.1A seemingly does not depend on the lack of basic knowledge by students, as they proved to know the same notions if asked in a different context or form.
- If the notion involved is included within a problem solving context, failure is mainly caused by the complexity of the problem and the presence of 'difficult' steps that draw students' attention.

It seems that students focus on the part of the statement they recognize as more important (related to the intentions of the writer) and interpret it according to standard mathematical semantics, with no application of CP. For example, the 2nd complex, non-real root is not mentioned in the statement but the students know, by ‘didactic contract’ that the theorems on complex roots of real polynomials are part of the curriculum and are likely to be required. The other parts of the statement seem to be interpreted according to conversational schemes, i.e. schemes based on common language which may apply CP less or more consciously. So discouraging students from applying conversational schemes should produce an improvement of results. This conjecture has been checked by comparison of problem 3.1A with a modified version, problem 3.1.B, which is equivalent, but presents a different wording. This procedure has been arranged for the sake of research and is not suggested as an instructional method at all. In other words, I am not suggesting that the use of natural language should be avoided, nor that teachers must choose particular wordings in order to make easier the interpretation by students.

Problem 3.1B: Find out, if possible, a polynomial with real coefficients, of degree 4, satisfying all of the following conditions:

- at least 1 of its roots is an integer;
- at least 2 of its roots are real;
- at least 1 of its roots is a complex non real number.

The two presentations are similar even as regards vocabulary. Version B has been written in order to discourage interpretations based on conversational schemes, by means of punctuation, layout and the repetition of part of the sentences (“at least … of its roots is/are …”). This could have marked the difference of the situation from a everyday-life one, so that students were prevented from applying CP.

4.2 Some Results

The two versions were given to 25 freshman computer science students, at the end of 1st semester, after taking some 30 hours of introductory algebra. 13 students took version A, 12 took version B. The two samples proved equivalent, as they performed almost identically in all the other problems of the test.

	Correct answer	*Wrong answer*	*No answer*
problem 3.1A	4 (31%)	7 (54%)	2 (15%)
problem 3.1B	7 (58%)	2 (17%)	3 (25%)

Tab. II: Answers to problem 3.1

These data are not conclusive. Further research is needed to corroborate the interpretation suggested. Nonetheless, the analysis of protocols shows that students with version B paid more attention to the interpretation of the data, by means of Venn diagrams or explicit remarks like '$\mathbf{Z} \subseteq \mathbf{R}$' and so on. Students with version A, including those providing a correct answer, generally did not pay much attention to the interpretation and the representation of the data.

After the test all the subjects were interviewed by senior mathematics majors. All the subjects giving a wrong answer ascribed their errors to lack of care. All the students have been shown both versions and have been asked if they believed that the presentation could have affected their answers. 21 students, out of 25, claimed that the wording of the statement of the problem did not affect their answers.

The outcome of the test and the interviews suggest two remarks:

- The wording of the problem did affect the answers to some extent.
- Students were not aware of this effect.

5. Discussion

The findings reported in sections 1, 2 and 3 show that the conversational interpretation of mathematical statements may account for a number of behaviors. But phenomena like these do not depend only upon language. They involve students' attitudes toward mathematics and mathematical tasks as well. Then I am not claiming that the misinterpretation of the texts of the problem is the only reason for the errors reported. The point is that often students do not use their mathematical knowledge in order to solve a problem. Some answers to problem 1.1 are examples of a behavior like that, as students do not use the obvious fact that, if n is an integer, '$n<100$' is true if and only if '$n\geq100$' is false. Another example is problem 3.1A, where the obvious fact '$\mathbf{Z}\subseteq\mathbf{R}$', which could easily avoid the misinterpretations reported in section 3, is overlooked. In other words, the use of knowledge should increase effectiveness but also stability of resolution procedures. More examples at this regard can be found in Ferrari (1996, 1997). Roughly speaking, it seems that students apply CP in some situations and their knowledge in others. The criteria they follow to choose whether to apply conversational schemes or more specific knowledge are related to what Vinner (1997) names *pseudo-analytical* behavior. Often students do not read the statement of the problem with the goal of recognizing the mathematical context they are asked to work within and to re-construct the knowledge that is required; they simply read the statement (or only a portion of it) in order to find verbal clues that could tell them which behavior (maybe out of pre-arranged set) they are expected to activate. This could explain why students' behavior depends upon the wording of the statement so strongly: some words and layouts suggest them to apply their mathematical knowledge (or pseudo-knowledge), activating a sort of script selected from their experience in mathematics. From this viewpoint, interpreting the purposes of the interlocutor becomes far more important than recognizing the mathematical ideas involved in the problem. This obviously amplifies the effects of the undue application of CP (or other improper interpretation schemes) to mathematical statements. I think that the link among linguistic obstacles and pseudo-analytical behavior is a fundamental one: should students actually read the statements with the purpose of understanding their meaning, a weaker dependence on the wording would be found. In most cases students possess some basic linguistic skill, but do not use them when reading the statement of a problem, as they search for verbal clues only, and so even a slight change may affect their interpretation.

I am not suggesting that ordinary language is a source of errors or may induce non-formal mathematical behavior. On the contrary, I think that the use of ordinary language when doing mathematics is unavoidable and a powerful tool to understand and communicate. But inaccuracy in the use of ordinary language, and the lack of knowledge of functions and usage of mathematical language (including symbolism but also ordinary statements when interpreted according to mathematical semantics) often go together with pseudo-analytical behavior. From the one hand, people working in the pseudo-analytical mode do not try to interpret statements according to their mathematical knowledge and thus it is natural for them to apply conversational schemes; from the other hand, people who cannot interpret mathematical statements reason are prevented from using their knowledge (if any) and compelled to search for verbal clues and to adopt pseudo-analytical behaviors.

These effects are increased by the practice of working within one representation system only, which make difficult to recognize equivalence or analogies between problem situations or to avoid obstacles. Sometimes students know different representations but cannot use or coordinate them (see Duval, 1995). Some answers to problem 2.1 are examples of lack of coordination of representation systems.

A thorough discussion on the instructional methods that could overcome these obstacles is out of the reach of this paper. Experience shows that students with good linguistic competence and accustomed to use language in scientific contexts too (not only to use language to show they master language) generally can easily overcome these obstacles. Working with multiple representations, with the help of technology, is another factor that can improve understanding. But any effort will be hopeless if we do not remove the factors which induces students to try to solve problems with no use of knowledge. Most likely it is neither possible nor desirable, to completely inhibit pseudo-analytical behaviors (for a discussion on this aspect see Vinner, 1997). But maybe it is possible to prevent them from hindering any further understanding. This requires deep changes in instruction and in assessment methods.

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THE ROLE OF WAYS OF INFORMATION CODING IN STUDENTS' INTELLECTUAL DEVELOPMENT

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Abstract: *Ways of information coding are subjective means with the help of which the surrounding world is reproduced in individual experience. Mastering mathematical concepts presupposes solution of two didactic tasks: firstly, including in the process of teaching three ways of information coding: verbal, visual and sensual - sensory with consideration of certain requirements to introduction of each of them; secondly, organisation of self-transference in the system of these three ways of information coding. Such activity in the process of work with concepts should lead to success of individual intellectual behaviour.*
Keywords: *ways of information coding, cognitive competence, individualization of intellectual activity.*

1. Introduction

Ways of information coding are subjective means with the help of which the surrounding world is reproduced in individual experience and which ensure organisation of this experience for future intellectual behaviour.

Principal ways of information coding were described in the works of George Bruner (1971, 1977). Bruner speaks of existence of three principal ways of subjective presentation of information: in the form of object actions, visual images and linguistic signs. Development of a child's thinking takes place in the course of a child's mastering these three ways of subjective information coding which are in relations of mutual influence and interaction. Analogous idea that the work of a thought is ensured by three "languages" of information processing - linguistic, image - bearing and tactile - kinaesthetic — was repeatedly expressed by L. M. Vekker (1976).

So, in our opinion, the informational exchange between a person and environment is ensured by three principal forms of experience: 1) in the form of verbal signs (verbal way of information coding); 2) in the form of images (visual way of information coding); 3) in the form of sensual impressions (sensory way of information coding), (Kholodnaya, 1997).

It is necessary to underline that the problem of the ways of information coding is not completely identical to the problem of forms and representations of information. Information may be represented to a person in various forms: by means of somebody's speech, by means of illustrations in a text-book and by a text itself, by means of computer ambience, by demonstration of real objects and so on. But any information, irrespective of its source, is transferred to individual subjective experience by three interacting psychic "channels". This basic cognitive mechanism, namely: mechanism of reversible mutual transference in the system of three ways of information coding influences two basic lines of a child's intellectual development. Its formation determines, firstly, growth of conceptual competence due to integration of different forms of experience and, secondly, growth of individualisation of intellectual activity due to revealing individual intellectual styles.

Within the frames of this paper the following questions of algebra teaching will be discussed:

1) in what form should verbal, visual, sensual ways of mathematical information coding be introduced, as well as how to organise the transference of experience from one form to another one;
2) in what way, on the basis of mutual transference of information in different forms of experience, can we ensure revealing and further forming of students' individual intellectual styles?

In our opinion, these questions may be solved by means of organisation of school texts. Such work was performed in the project "Mathematics. Psychology. Intelligence" in the frames of "enriching model" of teaching.

2. Ways of Information Coding as One of the Conditions of Growth of Conceptual Competence

Researches of conceptual thinking showed that the process of concept functioning presupposes simultaneous participation of three forms of experience — verbal, including the usage of words of the native language and artificial signs, visual and sensory. Accordingly the conclusion was made that conceptual thought is the result of reversible mutual transferences in the system of three "languages" of information processing (Vekker, 1976; Kholodnaya, 1983).

In this case, it is important, in our opinion, to underline the following. On the one hand, concept is presented as an unit of knowledge which exists objectively and which a child learns in the process of education. On the other hand, a concept is formed (made, summed up) within individual mental experience, acting as conceptual psychic structure.

Taking into account what has been said above we may schematically represent the process of forming conceptual structure in the following way:

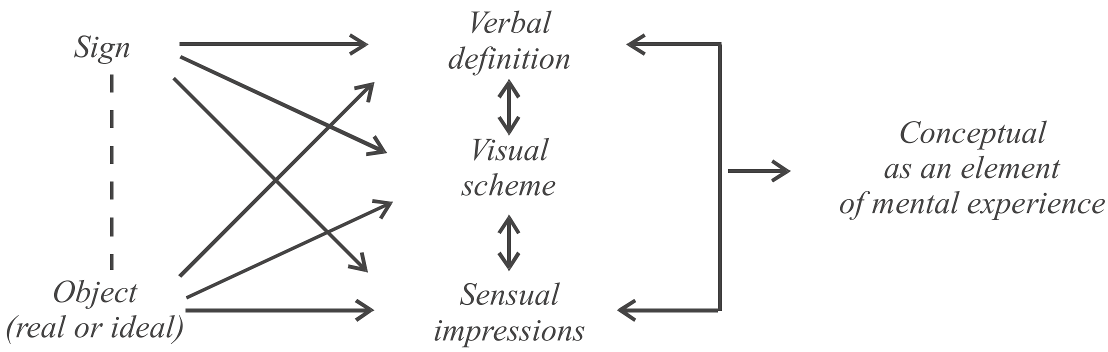


Fig. 1: Correlation of verbal, visual and sensory forms of experience in the process of concepts forming

So, when we comprehend anything at the conceptual level, we give it verbal definition, we have its image in our mind and we spontaneously feel it.

The problem of taking into consideration the psychological nature of the process of formation of concepts is rather important in educational process. Very often, traditional teaching, making the words (signs, formulas, symbolic expressions) nearly the only means of intellectual communication with a child, ignores the key meaning of the other two, equally important for development of intellectual abilities of children to accumulate knowledge about the world — through action and image. However, without usage and proper organisation of sensory (including active) as well as visual experience of a child, complete mastering of the meaning of signs and symbols (at the level of comprehension of concepts) becomes difficult. Language “codes” run idle, touching only superficial layers of a child’s ideas of the world.

So, forming concepts “inside” a child’s mental experience, that is growth of his/her conceptual competence, presupposes taking into account one of the basic cognitive mechanisms of intellectual development — mechanism of intercoordinated functioning of these three “languages” of information coding. Accordingly, mastering mathematical concepts requires solution of two didactic problems:

1) including in the process of education three ways of information coding — verbal, visual, sensual - sensory — with consideration of certain requirements to the introduction of each of them;
2) organisation of reversible mutual transference of information in the system of these three ways of information coding.

Verbal way of information coding may be introduced by active participation of students in formulating definitions, creating a formula, comparing different definitions and records of mathematical expressions. Students should also with the help of a word describe characteristics of objects and links between them.

Let us give an example of such work with the theme "Monomials". Students are given the following task:

Task 1. (Gelfman 1998, pp. 76-77).

Answer the following questions in writing:

a) How many months are in t years?
b) How many hours do n minutes make?
c) How many cubic centimetres are in m cubic metres?
d) How many minutes are in m days, in n hours?
e) What is the area of a figure made of three equal rectangles with sides a and b?
f) What is the volume of a body made of five equal parallelepipeds with edges a, b and a?
g) What is the area of a square with side c?

h) How many metres are in a kilometre?"

Task 2. (Gelfman 1998, p. 77).

Record the following algebraic expressions:

a) length of a circumference, an area of a circle of radius r ;
b) product of a variable of the 5th power x and a variable of the 4th power y , divided by six;
c) product of tripled product of a square of variable x and variable y by doubled product of cubes of the same variables.
d) product of variable x in the 5th power by variable y in the 4th power;
e) a half of the above-given expression;
f) the quadruplicated above-given expression.

After doing this task, students may check the results of the work of changing the natural language for the language of mathematics by means of the expressions which are written on the blackboard in a classroom:

50 ;	a^2b ;	$4x^3b^2y^4$;	$(-2)a^3x^2yc^4$;	$12t$;
x^5y^4 ;	c^2 ;	$3ab$;	$\frac{n}{60}$;	$5a^2b$;
$6x^5y^4$;	$60n$;	$24\cdot60m$;	$3x^2y\cdot2x^3y^3$;	10^3 ;
$25a^3bc\cdot(0,2)a^2cb^2$;		$\frac{1}{2}x^5y^4$;	$4\cdot\left(\frac{1}{2}x^5y^4\right)$;	
$4\frac{1}{2}a^2x\cdot(-4)axyc^4$;		πr^2 ;	$2\pi r$;	$10^6\cdot m$.

After this work, the students notice that all these expressions should get a new name as the students have not been confronted with such expressions before. The teacher says that all these expressions have one name in common "monomial". Then the students are given the following task:

Analyse the structure of all these expressions and try to define a monomial. What is your opinion, which of the following statements may be the definition of a monomial? Which of them can not be the definition of a monomial?

1. Product of some powers, the foundations of which are variables or numbers.
2. Algebraic expression, containing product of variables.
3. Algebraic expression, containing only operations of multiplication and raising to a power.
4. Algebraic expression, containing a variable.
5. Algebraic expression, which is a product, the multipliers of which may be numbers, one or a few variables, each of which is raised to a certain power.

6. Algebraic expression, which is a product, the multipliers of which may be powers of variables and numbers."

Doing this task, students should point out the characteristics of the notion "monomial" and, if they disagree with any definition, they should give their arguments.

Then we give some more tasks for students to work in more differentiative and variative ways with records of monomial and its variants. Two examples of such work are given below.

Task 3. (Gelfman 1998, p. 86).
Fill in the table

Monomial	Coefficient	Letters	Power of a monomial	Standard form	Example of a similar monomial
$5a^5b^5$	5	…	5	$5a^5b^5$	$-2a^5b^5$
$\frac{3}{4}ax\left(-\frac{5}{6}ax^3\right)$	…	a^2x^4	…	…	…
…	6	$x^5y^4z^3$	12	…	…
$\ldots a^3b(2,5)a^2b^2$	6	…	8	…	…

Task 4. (Gelfman 1998, p. 85).
Find the numerical values of monomials:

1) ab^2x ; 2) $0{,}5abxb$; 3) $10b^2ax$; 4) $-10axb^2$; 5) $-\frac{3}{4}ax\left(-\frac{5}{6}ax^3\right)\frac{1}{2}b^2ax$;

if a = 2 ; b = –3 ; x = 4 .
The task may be easily done, if you see, that …

It is very useful to give students ability to show their creative abilities in using verbal ways of information coding. So, working with the same theme "Monomials", we may give students the following task:

Task 5.
You are a TV-showman, and you should introduce Mr. Monomial to interviewers. Submit your questions and illustrations for the programme to the editor (a teacher).

Visual way of information coding presupposes, as one of the variants, usage of normative images. A table of orders, numerical ray, numerical axis, intercept, graph of a function, areas of figures and the like are referred to such images.

Special work should be conducted to make these images generalised and dynamic. So, for example, many teachers use the table of orders only for introduction of natural numbers but this image is not developed the future. But it is very useful to develop this image while studying operations with natural numbers and with decimal fractions. Then, the image will help students to be active in formulating rules of operations with decimal fractions, because he knows how to act in the table of orders to the left of decimal comma, so he would project the actions to the right of decimal comma. Here are examples of development of work with table of orders:

Task 6. (Gelfman 1997, p. 77).

Read and record by words and digits:

Thousands	Hundreds	Tens	Units	Decimal comma	Tenths
			6	**,**	**6**
					One plus nine tenths
			0	**,**	**4**
					One hundred plus seven tenths
1	**2**	**3**	**6**	**,**	**0**

Special role is given to individualised images of students themselves. So, speaking of the ways of solving rational equations, a student suggests using schematic representation of these images in the form of a bunch of keys (Fig. 2). Later we included the following picture of the bunch of keys in the text-book (Gelfman 1996b, p. 207).

One more example of visual way of information coding. In the process of looking for definitions of multiplication of different numbers, monomials, polynomials, it is very useful to give the task of defining areas of rectangles and squares. It is important for students to see development of these images in the process of learning different, but closely connected, products. Here is one of the tasks, which serves this purpose.

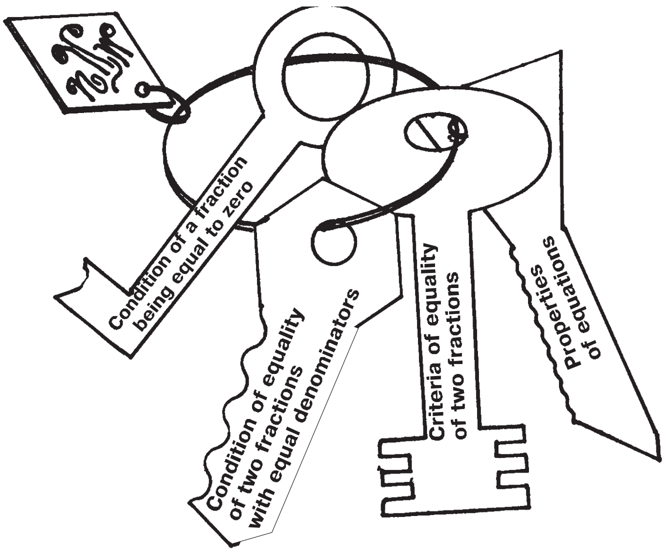
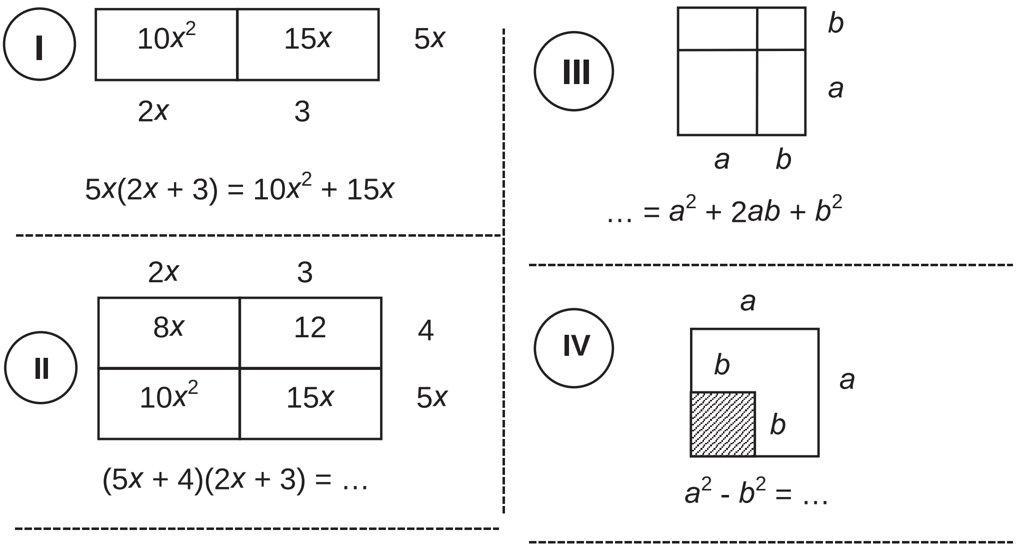


Fig. 2: Student's suggestion for schematic representation of the ways of solving rational equations

Task 7. (Gelfman 1996b, p. 91).

Look at the drawings and restore all the algebraic expressions which have been omitted in the recordings.



The same task is one of the examples of work for transition of one form of information coding into another one. Such work is necessary for forming any mathematical notion.

Sensual-sensory way of information coding may be introduced through objective-practical motivation, performing objective actions with some material objects, using sensual evaluations (bulky-compact, strict-chaotic, beautiful-ugly and so on). So, for example, to motivate usefulness of bringing monomial to standard form we say to students:

Pay attention to the fact that a standard form of a monomial is much simpler than the original one. How many, for example, different types of cards would you prepare for monomials, which we have got in the task?

3. Ways of Information Coding as One of Conditions of Individualisation of Intellectual Activity

In the process of learning mathematical concepts, different students have different correlation of three basic ways of information coding. One or another form of experience may be more predominant; to one student it is necessary to explain some material in words; to another student it is necessary to show and to a third one — to give the possibility to experience (including one's own actions). On this basis individual cognitive styles, by which we understand individualised ways of studying reality, are formed.

Formation of mechanism of reversible mutual transformation of three ways of information coding gives the possibility of organising an educational text in such a way as to represent ways of teaching students with different styles of cognition (different casts of mind).

In particular, predominance of verbal form of experience may find its expression in verbal-analytical, verbal-categorical, verbal-algorithmical cognitive styles, visual form of experience — in figurative-illustrative and figurative-modelling cognitive styles, sensory form of experience — in active-practical, associative-game and sensual-intuitive cognitive styles.

The work with students in our text-books is organised so that students with different cognitive styles could try different strategies of education which they could use.

So, for example, students were given the task to write the contents of a paragraph

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Students with verbal-analytical cognitive styles gave, for example, such tasks for this paragraph:

Raise to the third power binomials $a + b$ and $a - b$. Read the identities, you've got. Describe the right parts of the formulae: point out the number of monomials, their signs, coefficients, exponents.

Students with verbal-categorical cognitive styles suggested carrying out an investigation:

"You already know the identities

$$(a + b)^2 = a^2 + 2ab + b^2;\ (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Can you open the brackets in the expression $(a + b)^4$? What do you think, how many members will be in the expression you've got, how will the exponents work, what will be values of the coefficients?" "Get formula of transformation of cube of trinomial $(a + b + c)^3$ into multinomial."

Tasks of students with verbal-algorithmic cognitive style were like that:

"Compose algorithm of getting multinomial from cube of sum", "Consider two solutions and mark basic steps in them".

Tasks of the type:

"Try to compose domino, lotto, labyrinth, in which identities
$(a \pm b)^3 = a^3 \pm 3a^2b \pm 3ab^2 \pm b^3$ may be used",
"Spot the mistakes …"
were suggested by students with associative-game cognitive styles.

Students with figurative-illustrative style gave such tasks:

"Look at the drawing … . Explain it with the help of formula …";
"Make up a scheme of formula of cube of binomial".
Students with verbal-demonstrative cognitive styles worked with such tasks:
"Prove the identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ ";
"Fill in the blanks in the identities …"; "Complete the expressions so that they may be transformed with the help of formulae …".

Students with sensual-intuitive cognitive style gave the task:

Consider tasks … Try to make an advertising slogan in the name of the task itself. For example, "Meeting the situations in which it's expedient to use identities".

4. Conclusion

Intellectual up-bringing of a child in the process of school education presupposes, on the one hand, raising productivity of his/her intellectual abilities (in particular, growth of cognitive competence, which is directly related to success of individual intellectual behaviour. On the other hand, strengthening individual originality of his/her cast of mind (in particular, formation of individual cognitive style, which provides effectiveness of individual intellectual adaptation to the requirements of environment). Solution of these two tasks is connected with working out comparatively new technologies of teaching, oriented at actualisation and enrichment of basic components of mental experience of every child (including the level of his/her cognitive, metacognitive and intentional experience). Within the frames of this paper we tried to discuss possible variants of consideration of one of components of cognitive experience, namely: ways of information coding. It allows, in our opinion, to raise the quality of students' mathematical knowledge and to individualise the process of teaching mathematics.

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ON TEACHING STUDENTS TO WORK WITH TEXT BOOKS

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Abstract: *Ability of using different ways of work with a text-book is one of the most important intellectual characteristics of student's personality. It may be formed by a teacher and by a student himself by means of specially designed tasks.*
Keywords: *-*

To make a student successful in his mathematics studies a student should not only have some particular skills but some general skills as well. One of the most important skills is the skill of information processing, in particular, to work with educational literature. An outstanding Russian mathematician V.M. Glushkov writes: "There is the necessity of information processing in every sphere of human activity. Namely, a translator, an economist, a mathematician and, even, a poet has to process information. Information processing is what we call human intellectual activity".

The problems of work with school-texts are treated in the works of Bell (1987), Granik (1988), Getsov (1989) and others.

The group of the authors of the project "Mathematics. Psychology. Intelligence" (headed by Prof. E. Gelfman), wrote a series of text-books for students, aged 10-15. The main idea is that a text-book should perform a role of an intellectual self-instructor, where a student works directly with a text-book and that gives him the possibility of finding the most suitable and convenient for him way of mastering school material. Such an approach presupposes that a student should be, first of all, taught how to use a text-book and this instruction (in this or that form) is given in every text-book of our series.

First of all, the books themselves teach students different ways of information processing. They may show to students, how, for example, one and the same material may be represented as a story, a dialogue, a glossary, a system of tasks.

Our text-books also teach students, aged 13-15, how to use reference books. Every text-book is supplied with special sections, entitled “Glossary”, “Reference book”, “Samples of solving problems”. There are also special tasks which stimulate the work with reference-books and glossaries, samples of problem-solving not only from a given textbook. Students are given tasks to make some entries into glossaries and reference-books, to make their own samples of problem-solving. Here are two examples of such type of work:

How do you usually act when you are confronted with a new problem? Do you try to find the solution independently? Do you ask help from a teacher or a friend? Do you read special books? Do you turn to reference books? Let’s try to turn to reference book on mathematics. Let’s take, for example, “Reference book on mathematics”, written by A.G. Tsipkin. We shall look for the problem requires in the table of contents and we shall look into the index of problems. Though, we don’t know yet, what word should be looked for in this index. First of all let’s turn to the table of contents. These is point 5.3, which is entitled “Quadratic equation”. Here the equation $ax^2 + bx + c = 0$ is given, which is called quadratic. Equation $2x^2 – 53x + 300 = 0$ looks very much like it. We have only to realise that $a = 2$, $b = –53$, $c = 300$. So, what is said in the reference book about solution of such an equation? … . (Gelfman, 1997).

Reduce fractions to a common denominator:

a) $\frac{x}{x+y}$ and $\frac{y}{2x}$ b) $\frac{a}{a^2-1}$; $\frac{b}{a+1}$; $\frac{c}{a-1}$;

c) $\frac{m}{a^2+2a+1}$; $\frac{n}{a^2-2a+1}$; $\frac{p}{a^2-1}$;

d) $\frac{c^2}{12ab^2}$ and $\frac{a^5}{8b^3c^2}$; e) $\frac{4}{a-b}$ and $\frac{1}{b-a}$.

Try to formulate the rule for reducing algebraic fractions to a common denominator. If you have problems, turn to a reference-book. (Gelfman, 1996a).

We give students tasks to write summaries of a text, make advertisements and anti-advertisements of some methods of solution of problems of a certain type or of some notions. We also teach them how to write synopses, notes, how to present historical data or how to construct a text. Here are some examples.

Adding of Integers

with identical signs	with different signs and different moduluses:	with different signs and identical moduluses:	with a zero:
3 + 2 = 5; –5 + (–2) = –7	–5 + 2 = –3; +5 + (–2) = 3	–5 + 5 = 0	–5 + 0 = –5; 0 + 1 = 1
Common sign and the sum of moduluses	Sign of a “strong” summand and difference of moduluses	Just simply we get a zero	

Using Malbina’s abstract answer the question:

1. In what cases you have to:

a) to add natural numbers

b) to subtract natural numbers for getting the sum of integers?

2. What number should be added to –5 to get 0; –5; +5 ? (Gelfman, 1996b).

Write a short composition on one of the problems given below:
“Application of Viet’s theorem”, “Assumptions which may be driven from Viet’s theorem”, “Roots of quadratic equation and Viet’s theorem”, “What new material have I learnt thanks to Viet’s theorem?”, “Around Viet’s theorem”. (Gelfman, 1997).

The following forms of work with texts are given in our textbooks:

1. Page by page text analysis
2. Making plans of texts
3. Making up questions to a text
4. Reviewing
5. Foretelling the contents of a paragraph by its title
6. Conducting role-games
7. Assessment of the tasks given in a text-book
8. Rendering a text from a student’s or a character’s name

and so on.

Sometimes we devote the entire period to work with books: having such classes as: conferences, consultations, competitions for the best lecture, text or story.

As our experience shows such work at mathematics lessons helps students to be more successful not only in mathematics but in other school subjects as well.

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PROJECTS AND MATHEMATICAL PUZZLES - A TOOL FOR DEVELOPMENT OF MATHEMATICAL THINKING

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Abstract: *The teaching of mathematics in the Czech Republic has traditionally been of an 'instructive' nature. This teaching strategy has alienated most students in mathematics because they have been expected to use the skills they have learnt without understanding the underlying concepts. They have not been able to appreciate the usefulness of mathematics nor get enjoyment from the subject. From our own teaching experience, by using projects and mathematical puzzles we have found that our students have gained the necessary understanding, enjoyed their work and developed other important attributes such as the ability to conjecture, to work systematically and to communicate. In this paper we set out the benefits of using this practical approach to the teaching of mathematics together with an analysis of the processes necessary to set up such teaching strategies in schools etc.*
Keywords: *investigative methods, understanding.*

1. Introduction

The prime source of our interest for the use of investigative teaching methods in mathematics education was the long-term experience of one of the authors with her own direct teaching of mathematics, whom did not fit the traditional concept of teaching in Czech schools and the experience of our English colleague who has been using investigative approaches in school and university since the early 1960s. This conception is based on

- the teaching of school subjects separately (even of individual fields such as algebra or geometry in mathematics),
- instructive delivery of complete knowledge,
- repetitive practice of skills.

The main consequences of it are:

- The emphasis in teaching is put on the content, not on the student's learning.
- Knowledge and skills gained by most students are not permanent and students are not ready to use them and to find new or missing information.
- Students are unable to use their mathematical skills in informal examples and problems.
- Students are often afraid of mathematics because of the abstract way in which it is taught; they are stressed and consequently their performances do not correspond with their abilities.
- Students (particularly the less-able ones) never have the possibility to feel the pleasure of "discovering" something new and of achieving their own potential.
- Students are not able to plan, develop strategies, work independently or react flexibly in concrete situations.
- Students are lacking in creativity, flexibility, critical thinking, the ability to reason meaningfully, the ability to think procedurally, to understand multi-causality.
- Students lack initiative, self-reflection, self-confidence, and ability to overcome obstacles.
- Students' skills work co-operatively in groups and to communicate the results of their work are not developed.
- The empathy is not developed.
- There is a lack of enjoyment in mathematics.

2. Our Approach to Projects

We chose an investigative teaching strategy because it is one of the possible ways of eliminating the problems that the traditional authoritative and instructive teaching produces for a student. It was mainly this method's naturalness with its emphasis focussed on the student and the complexity within projects that made us use it. In full agreement with (Grecmanova & Urbanovska, 1997), (Littler & Koman, 1998), when preparing/using investigative teaching we take advantage of the following facts:

1. The student inclines to relate all his/her activities with his/her specific need or function in every moment.
2. The student's understanding of reality is gathered in natural way from his/her autonomous experiences, by direct contact with the world of mathematics, in the form of experimenting and investigating . This is what J.A. Comenius has already stated (Comenius, 1631): "It is necessary to proceed from concrete, because it is the only item that is familiar to the student."
3. The student does not learn already complete and/or abstract knowledge by means of algorithms, but constructs this knowledge and during any solving procedure he/she shapes their knowledge to enable the various parts to connect. Therefore, as a necessity during the projects construction, we put great emphasis on the following demand: each project should offer the student a real problem in a relevant context, a challenge for them to create or construct their personal strategies for solutions and understanding of mathematics. For us it means that our student's learning processes become cognitive instead of only being pedagogic.
4. The student is able to develop his/her cognitive abilities by practical experience and cognition being the basis of understanding. In a non-instructive way we teach the student something that we consider to be more important than skill acquisition itself, namely methods of cognition.
5. The student's free choice of direction of cognition and better opportunities to participate in the project processes are strong motivating factors. The 'real life' possibilities of making a true or false conjectures add to the student's motivation and develop perservence.

6. Asking the student to work beyond their ability does not occur and his/her individual needs are prerequisite.

3. What Is a Project

A general definition of the concept of project/investigative teaching is not given here. There is no agreement on a definition in literature. Different approaches to the definition and classification have been occurring in literature from the very beginning of investigative teaching (Dewey, Kilpatrick or Vrána, Příhoda in Czech books) until today. At present investigative teaching strategies are having a renaissance (IHME, 1996), (Littler & Koman, 1998), (Koman & Ticha, 1997).

One of the definitions which most closely agrees with our thinking is that written specifically for teaching of mathematics and given in (Kubinova, 1997):

> *A project in mathematical education* means any independent work produced by a student or a group of students that leads to the active solving of a problem connected with a mathematical concept or mastering a skill. Its main feature is that a student can decide independently how and in which order he/she will solve the tasks necessary to successfully cope with the project.

In our mathematics teaching we distinguish three basic types of projects: *inter-disciplinary*, *inner-disciplinary* and *mathematical puzzles* (tab. I).

<table>
<tr><td rowspan="2">1a)

Inter-disciplinary projects from different disciplines.</td><td rowspan="2">1b)

Inter-disciplinary projects from one discipline.</td><td>2)
Mathematical puzzles</td></tr>
<tr><td>Developing logical thinking, strategies for winning or ‘not losing’, giving repetitive work of weak process.</td></tr>
<tr><td colspan="2">The student experiments, discovers, conjectures, proves, verifies, … relations with the project.</td><td>Clarity of aims and final goal.</td></tr>
</table>

Tab. I

Note: In 1, the projects can often be attempted by students of differing mathematical ability. Their final work which is presented should indicate the level of their mathematical thinking (Kubinova, 1997), (Littler & Koman 1998).

Examples:

1a)

- Dandelion seeds are scattered by the wind. Devise a long-term experiment (over the spring semester) which would provide you with data to show in which direction you are likely to find next years flowers.
- In one of his stories, Jack London describes his trip by sledge pulled by seven dogs from Skagway to the camp. The first day, the sledge run at full speed, but the next day, some dogs ran away with a pack of wolves. London continued the trip with the remaining dogs. Therefore, he reached the camp later than he expected. The author adds: “If the missing dogs had pulled 50 miles more, I would have been only one day late!” The task is to: a) calculate the distance between Skagway and the camp, b) find out more detailed information about Jack London.

1b)

- 36 squares are arranged into rectangles in as many ways as possible, using all the squares each time. Record the length and width of each rectangle you find. Plot these on a graph and find the relationship between the points. Which rectangle has the smallest perimeter and why is this?
- You know how to construct regular hexagon, octagon as well as square and equilateral triangle in a plane. Try to describe regular solids in space, draw their nets and make paper models of them. You can compare your description with the description in mathematical literature.

2)

- In the game of ‘noughts and crosses’ you play with a partner on a 3 x 3 board and to win you must have either three ‘noughts’ or three ‘crosses’ in a line. Devise strategies for not losing (a) if you start the game, (b) if you are second. How many possibly winning lines are there on a 3 x 3 board. How many winning lines are there if you are playing on a 4 x 4 board, a 5 x 5 board etc. Could you find how many winning lines there were on an n x n board?

- There are five numbered cubes in an open box. The box has room for just six such cubes, so there is a vacant space (Fig. 1) which allows the cubes to be moved around. They may not be taken out of the box in the course of the puzzle, only sliding movements can be made. Find the least number of moves which can transform the position in Fig. 2 into the position in Fig. 1.

	1	2
5	4	3

Fig. 1

3	2	4
5		1

Fig. 2

Both projects and puzzles enable the student to:
a) develop a student's strategic thinking and the ability to conjecture,
b) discover new relationships,
c) consolidate his/her knowledge and skills,
d) exercise hintertho imperfectly mastered skills.

We include all three types of tasks in our teaching on purpose, since it enables us to develop mathematical thinking in our students

- of different kinds (strategic, functional, algorithmic, combinatorial, …),
- on different levels according to their individual abilities.

That are very often puzzles can be used to show pupils the enjoyment of mathematics much more easily than any other way. They think they are playing games and this is especially the case if the pupils have had bad experiences with mathematics or are not confident.

3.1 Work With Projects

At present we proceed in projects mostly according to the following scheme that we however do not consider either to be definitive, or to be exhaustive.

PROJECT PREPARATION

setting of

aims & objectives	*What are teaching objectives?* *What learning objectives do you want to cover?*
choice of topic	*Choose an appropriate topic to meet above aims & objectives and to be in a context relevant to the student.*
scope of topic	*Does the topic cover* *several subjects?* *mathematics only?*
time allocation	*In school only?* *In school + in student's own time?*
topic mapping	*Several subjects:* *List areas for each subject, look for links, develop ideas.* *Mathematics only:* *What is the mathematics needed; the background of students; the links across different aspects of mathematics?*
resource needs	*Gather practical equipment needed, reference and source books.*
type of topic	*Open – teacher describes investigation* *Closed – teacher will need to prepare detailed brief of what has to be done*

PROJECT IMPLEMENTATION

- Open
  - preparatory stage
    - *Teacher to discuss the subject of the topic with students, engender enthusiasm, get them to put forward ideas.*
      Do they want to work individually, in groups?
  - implementation
    - *Students to set out a brief of what they intend to do and what they hope will be their outcomes.*
      Students experiment, teacher to act as facilitator only.
- Closed
  - preparatory stage
    - *Teacher to provide materials for experiment with brief giving details of what and how they are expected to experiment, after discussion of topic.*
  - implementation
    - *Students to work on their experiment in groups/individually as directed by the teacher.*

EVALUATION OF PROJECT RESULTS

Teacher

Were the projects defined clearly enough for the pupils?

Was the mathematics needed at a suitable level for the pupils?

Did I use a suitable teaching strategy?

Did the pupils meet all the objectives I had set?

Would I change anything when I do it again?

Did the students understand deeply enough the concepts used?

Student

direct

Students to report back verbally to class on their findings giving a brief presentation.

Teacher to have noted by observation the contribution made by individuals during the project and the final presentation.

Any outside observer.

indirect

Assessment of the student's written work to see whether they understood the problem, had the ability to systematise their experiment and could 'solve' the project.

Were they able to describe what they had found out?

3.2 Work With Our Tasks

In mathematics lessons our students work with project/puzzles individually or in groups. They use different solving strategies. The main ones are (Schuh, 1968):

- *Trial and error*. This activity is important and useful when solving a pure puzzle. Its disadvantage is, that it can invite the belief that solving a pure puzzle is a matter of accident, i.e. of luck. This opinion is quite widespread, but incorrect. Finding one solution when solving a pure puzzle, the solver cannot know whether he/she has found all solutions or not.
- *Systematic trial*. This is a trial where the solver knows precisely which case has been considered and which has not. The solver is following a logical sequence of cases defined mainly by their mathematical knowledge.
- *Division into cases*. The solver divides various possibilities into groups according to his/her own classification system. He/she works with so established sub-cases, sub-sub-cases, … .
- *Puzzle tree*. The procedure of grouping various possibilities can be used in the form of puzzle tree (example see Fig. 3).

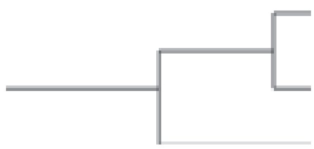

Fig. 3: Example of a puzzle tree

4. Conclusions

Implementation of projects in actual teaching was associated for us with several difficulties. On several occasions we had to face our colleagues' lack of understanding of the benefits and processes involved in teaching/using projects. But for us, it is a fundamental fact that using projects brought such changes in mathematics teaching/learning enabling our students to obtain positive attitudes towards mathematics and even the weakest students becoming gradually involved in solving projects.

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Acknowledgment

The paper is a part of a project GACR 406/96/1186: Mathematical education of 5 to 15-year-old students. The presentation was supported by The Ministry of Education of Czech Republic (Project PG 98374).

THREE-TERM COMPARISONS

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Abstract: *Based on previous work by Novotna et al., 12 three-term multiple comparisons problems were analyzed and field tested with teachers (in Israel) and students (in the Czech Republic). The variables in the study were: The compared and the reference in problems, the use of the verbal expression "more than" and "less than", and the underlying schemes. First analysis and the preliminary results are now presented here.*
Keywords: *multiplicative compare problems.*

1. The Problems

Here is a story (situation): Peter, David and Jirka together own 198 marbles: of which David has 22 marbles, Jirka has 44 and Peter has 132. This situation is the basis for several algebra problems. Novotna, for example used it under three different verbal forms and studied how six-graders solve them (Novotna, 1997). Here are the forms used by Novotna (this paper is based on the work Novotna started with Prof. Bednarz and Prof. Janvier at CIRADE and UQAM at Montreal (Bednarz & Janvier, 1994; Kubinova et al., 1994):

A1. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 6 times more marbles than David, and Jirka has twice more than David. How many marbles does each child have?*
A2. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 3 times more than Jirka, and Jirka has twice more than David. How many marbles does each child have?*
A3. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 6 times more than David, and 3 times more than Jirka. How many marbles does each child have?*

In her study Novotna found that though the same situation is described, the students performances are different according to the form used to present these three problems. Our analysis led us to formulate another nine possibilities for the same underlying situation. The next three problems are similar to Novotna's problems yet they describe the relationship between the boys, this time in terms of a "less than" relation. The other six problem mention both relations "more than" and "less than" in the same problems.

B1. *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has 6 times less than Peter, and he has twice less marbles than Jirka. How many marbles does each child have?*

B2. *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has twice less than Jirka and Jirka has 3 times less than Peter. How many marbles does each child have?*

B3. *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has 6 times less than Peter, and Jirka has 3 times less than Peter. How many marbles does each child have?*

C1. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 6 times more than David, and David has twice less than Jirka. How many marbles does each child have?*

C2. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Peter has 3 times more than Jirka, and David has twice less than Jirka. How many marbles does each child have?*

C3. *Peter, David and Jirka play marbles. They have 198 marbles altogether. David has 6 times less than Peter, and Peter has 3 times more marbles than Jirka. How many marbles does each child have?*

E1. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Jirka has twice more marbles than David and David has 6 times less than Peter. How many marbles does each child have?*

E2. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Jirka has 3 times less marbles than Peter, and twice more marbles than David. How many marbles does each child have?*

E3. *Peter, David and Jirka play marbles. They have 198 marbles altogether. Jirka has 3 times less marbles than Peter, and Peter has 6 times more than David. How many marbles does each child have?*

2. Analysis of the Problems

All the above problems are of the type known to be "multiplicative compare problems". As already mentioned (Nesher, Greeno, & Riley, 1982) the compare problems are analyzed in terms of a two-place relation R(a,b) in which 'a' is the compared quantity, and 'b' is the reference quantity. The comparison relationship can be described by either "more than" or "less than". One should note that the relation "more(a,b)" describes the same relationship as "less(b,a)", in which the arguments changed role, the compared becoming the reference and the reference becoming the compared quantity. To say "David has less than Peter" is the same as saying "Peter has more than David". The comparison relation is asymmetrical and in most cases we have the choice as to the wording to describe linguistically the situation; that means we may decide what will be for us the compared quantity, and what will serve as the reference. The compared quantity will then be the subject in the sentence that describes the situation, and the reference will be part of the predicate.

From this point of view, each of the above 12 problems, involves three comparison relations:

(a) The non-ordered (the non-ordered relation between David and Peter, is going to be an ordered relation once we use the verbal description "more" or "less") comparison between David and Peter R(D,P), or R(P,D).

(b) The non-ordered comparison between David and Jirka R(D,J), or R(J,D),

(c) The non-ordered comparison between Peter and Jirka R(P,J), or R(J,P),

where P, J and D stand for Peter, Jirka and David, respectively.

In the above problems (A1-D3) the total amount of marbles is given. This sows that in order to present a problem that can be solved, only two relationships of comparison are needed. There are altogether three different combinations of two relations. Therefore, in all the above problems we will find one of the following combinations of the basic comparisons:

I. The problem mentions the non-ordered comparison between R(D,P) and R(D,J). (Problems: **A1, B1, C1, E1**)

II. The problem mentions the comparison between R(D,J) and R(P,J). (Problems: **A2, B2, C2, E2**)

III. The problem mentions the comparison between R(D,P) and R(P,J). (Problems: **A3, B3, C3, E3**).

As explained before, each of the comparison relationship is asymmetrical. Then, with the use of "more than" and "less than" for each of the basic relations (a-c), we arrive at 12 different formulations of word problems for the **same** underlying structure, for the same situation described in the beginning of this paper.

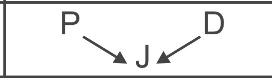
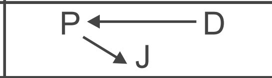
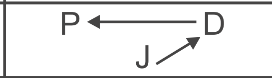
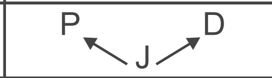
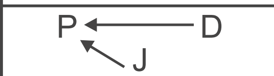
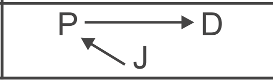
3. Variables Involved in the Study

3.1 The Compared and the Reference

As we have already mentioned, each problem consists of two explicit compare relations (see, I - III). Yet, the direction of the description and the wording used in the comparison are different and they will be distinguished. For example, in A1 the relation "more than" is employed in both relations. Thus, the situation dictates that David will be the reference and Peter and Jirka the compared. Thus, in this problem two (Peter and Jirka) are the compared to one reference (David). In problem B2, although it is again a type-II problem as explained before, here, because of employing the relation "less than" it is David, the compared and Jirka and Peter are the reference.

The choice of the "compared" or the "reference" dictates which will appear as the subject of the sentence (in the word problem) and which will be part of the predicate. Table I describes the compare-reference situation in each problem the head of the arrow points to the reference.

From this point of view we could distinguish three types of problems, those with one compared and two references (we mark it "1 to 2") (problems: A3, B1, E2), those with two compared and one reference ("2 to 1") (problems A1, B3, B2), and those with 2 compared and 2 references ("2 to 2") (problems: A2, B2, C1, C3, E1, E3).

A1		2:1	M	C1		2:2	ML
A2		2:2	M	C2		2:1	ML
A3		1:2	M	C3		2:2	ML
B1		1:2	L	E1		2:2	ML
B2		2:2	L	E2		1:2	ML
B3		2:1	L	E3		2:2	ML

Tab. I: Analysis of the problems

From the syntactic point of view, the “2 to 2” format problem text are a conjunction of two full sentences. The predicate of the first sentence becomes the subject of the second sentence. These problems seem to be the easiest ones to grasp from the point of view of the flaw of information. Problems of the “2 to 1” format, are a conjunction of predicates, and as those of the “1 to 2” format, they make an anaphoric mention of the same subject which plays a role in the two comparisons.

3.2 “More Than” or “Less Than”

Several studies suggest that the word “more” is more easily comprehended than the word “less” (Donaldson & Balfour, 1968; Nesher & Teubal, 1975). In our case regarding three of the problems (A1, A2, A3), we have use the word “more”. Then in three problems (B1, B2, B3) we have used the word “less” and as for problem (C1-E3) we have used both concepts together.

At this point, we need to add a special comment. In English both the words “more” and “less” may describe an additive comparison (such as “Ron has 5 stamps more than Rene”, or Rene has 5 stamps less than Ron). For multiplicative comparison, however, the expression used is : “Ron has 5 times stamps as Rene”. The words “more” and “less” do not usually appear in a multiplicative comparison. But, in our languages

(Hebrew and Czech) the words “more” and “less” are used for the multiplicative compare relation as well as for the additive compare relation (Nesher, 1988).

3.3 The Underlying Scheme

In earlier studies by (Nesher & Hershkovitz, 1994) (Hershkovitz & Nesher, 1996) we have demonstrated the role of schemes in explaining the variance in students performance. The schemes were of three types:

(1) Hierarchical,

(2) Sharing parts, and

(3) Sharing whole.

Figure 1 presents these three schemes.

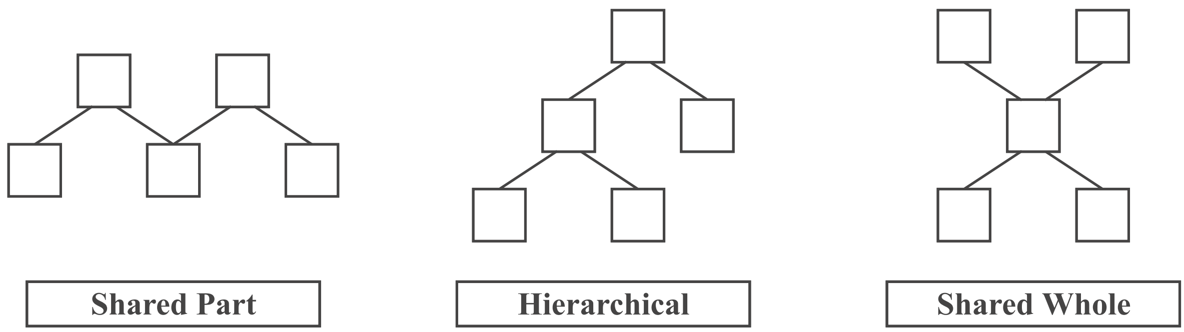


Fig. 1: The three schemes

An analysis of Novotna’s problems (E3, A2 and A3) has shown that each one of them belongs to a different scheme type. All share the fact that the quantity the children have altogether are 198 and we will not include it now in our presentation which is distinct for each problem.

Along this analysis we can map the set of 12 problems as follows:
The Shared-part scheme: Problems: A1, B1, C1, E1.
The Hierarchical scheme: Problems A2, B2, C2, E2.
The shared-whole scheme: Problems A3, B3, C3, E3.

To conclude:
Following Novotna studies the set of 12 problems, lends itself to a study of the role of each of the above variables. The 12 problems were given to experienced teachers as well as to students.

3.4 The Variables of the Study:

1) The scheme: Shared-Part (SP), Hierarchical (H), or Shared-Whole (SW).
2) The use of the relation "more than (M); "less than"(L), both (ML).
3) The syntactic combination of "Compared" and the "Reference" in the compound problems ("1 to 2", "2 to 1" and "2 to 2").

4. The Experiment

The set of twelve problems was given to about 100 teachers in an in-service workshop in Israel and to students in the Czech Republic. The teachers were all experienced in teaching mathematics in primary schools. The twelve problems were arranged in three separate forms by type, each containing only four problems from different classes. Each teacher solved only one form. The forms were distributed at random.

Each problem was solved by about 35 teachers. It took about 40 minutes for to complete the task.

5. Findings

Table II presents the raw data in percentage obtained for each problem by the teachers.

Problem #	**A1**	**A2**	**A3**	**B1**	**B2**	**B3**	**C1**	**C2**	**C3**	**E1**	**E2**	**E3**
%s success	100	86	74	67	78	92	86	81	76	76	58	80

Tab. II: Percentage of Success for Each Problem

More statistical elaboration concerning the role of each variable will be presented at the conference.

5.1 Strategies

The following strategies were observed:

I. Algebraic methods

II. Arithmetic methods

Within the algebraic methods, teachers used one variable but they each assigned a different quantity to the reference in the equations.

The references were:

I.1: Choosing the smallest quantity (D) consistently to be the reference in any problem they solved.

I.2: Choosing the smallest quantity (D) to be the reference when the comparisons were given by the word “more” and choosing the largest quantity (P) to be the reference when the comparisons were given by the word “less”.

I.3: Czech students used equations with two unknowns.

I.4: We could not find consistency.

II. The arithmetic strategy, consisted of:

II.1. Using a ratio strategy.

II.2. Division by the number of participants (used by Czech students).

Results from problem A3 solved in the Czech Republic in 1994, are presented in table III.

Grade	4th	5th	6th	7th
No. of solver	50	68	241	20
Successful	1	9	30	12
Unsuccessful	49	59	211	8
Arithmetical (Ar1)	9	17	51	0
Arithmetical (Ar2)	7	6	62	1
Algebraic (1 equation)	4	15	3	17
Algebraic (2 equations)	1	1	0	1
Algebraic (3 equations)	2	2	2	0
Non-scholar	2	1	0	0
Not solved by	25	26	123	2

Tab. III: Results from the Czech Republic

Ar1: Calculated using 1 part (David) (198:9=22 ; 22X6=132 ; 22X2=44)

Ar2: Calculated using number of participants (198:3= 66) then trying to satisfy the given relationship.

Note: Unsuccessful solutions were mostly probably influenced by the following reason: There was another problem in the set, where two participants were present and where the strategy was successful (The total fee which Mr. Novak's daughters, Pavla and Marie, received was 181 CZK. Marie had 37 CZK more then Pavla. How many crowns did each daughter receive?)

Algebraic, 1 equation (David x, Peter 6x, Jirka 6x : 2 = 3x) and modifications
Algebraic, 2 equations (David x, Peter 6x, Jirka y, 6x = 3y) and modifications
Algebraic, 3 equations (David x, Peter z, Jirka y, z = 3y, z = 6x) and modifications

Non-scholar: use of "approximation": the student estimated David's amount, calculated Jirka's and Peter's amount following the assigned relationship, then he/she added the three amounts and compared the sum with the assigned sum of marbles. In case of error in the sum, he/she restarted with another starting number.

Note: This strategy we found rarely used, but after using it once, the student used it systematically to solve all the assigned problems in the set.

Preliminary analysis indicates that some of the analyzed before variables are dependent. For example, the choice of reference is dependent on the problem's formulation, whether it uses the words "more" or "less". It seems that the compared and the reference in the problem were of greatest importance influencing the solver. About 90% of the teachers used algebraic strategies and only about 10% used arithmetic strategies. The most frequent algebraic strategy reported was the one where David was the reference, even when the text used others names as the reference.

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Acknowledgment

The paper is a part of a project Charles University Grant Agency: Analyza kognitivnich procesu zaku pri reseni slovnich uloh. The presentation was supported by The Ministry of Education of Czech Republic (Project PG 98374).

TEACHER STUDENTS' RESEARCHES ON COGNITIVE ACTIVITY WITH TECHNOLOGIES

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Abstract: *This paper represents the results of special courses given to undergraduate teacher students of «mathematics-computer science» speciality. A general idea that integrated these courses, consist in the training of future teachers for understanding the possibilities and limits of the use of technologies (systems like DERIVE and Cabri-geometre) for support of pedagogical control of learners' cognitive activity. It is very important to develop teacher students' abilities for their researches on why and how they should use technologies particularly like microworlds and computer algebra systems. It gave an opportunity to encourage teacher students' interest in teaching algorithmic and half-algorithmic problems as well as heuristic problems and therefore heuristic methods of operations that intrinsic of creative activities.*
Keywords: *cognitive approach, mathematical thinking and learning, teacher studies.*

1. Introduction

Modern education is oriented towards making students' creative abilities more active. This necessitates psychological and pedagogical researches devoted to the development of teacher students' creative capacities. To solve this complex problem, it is not sufficient to give students firm knowledge in theory of pedagogy, psychology, mathematics, etc., but it is necessary to mould and develop their creative features essential for independent researches.

This paper represents the results of special courses given to undergraduate teacher students of «mathematics-computer science» speciality. A general idea that integrated these courses, consist in the training of future teachers for understanding the possibilities and limits of the use of technologies (systems like DERIVE and Cabri-geometre) for learners' cognitive activity. It is very important to develop teacher students' abilities for their researches on why and how they should use technologies particularly like microworlds and computer algebra systems (CAS).

Since 1992 a special course "The use of mathematical packages for students explorations in Mathematical Analysis" has been worked out for undergraduate teacher students in their future classes for discovering and forming mathematical concepts. There is no doubt that teaching based on computer explorations contributes to the growth cognitive activity, but in non-computer variant it was out limits for the majority of teacher students.

But with the advent of improvement technologies they could feel its advantages and realize its essence. It gave an opportunity to encourage teacher students' interest in problems of pedagogical control of pupils' cognitive activity. It is necessary to modify programs and standards of the mathematics teachers training that concern using of technologies in their future pedagogical activity since modern software can promote modification of view on the essence of a pedagogical activity. In expectation of outcomes of long-term pedagogical investigations the inaction is short-sighted and inefficient therefore future teachers have to get preparation for their own pedagogical researches.

2. Background

Our previous researches (Rakov & Oleinik et. al. 1994, Oleinik et. al. 1996) has shown that it is insufficient for a student to have good knowledge, skills and habits. It is necessary to develop a person's psychological peculiarities essential for a certain activity and contributing to its successful fulfillment. They manifest themselves in a person's intellectual qualities system: the ability generate new ideas and put forward problems independently, flexibility and originality of thinking (the ability to perceive a well-known problem in new context), the ability to transpose knowledge and skills in a new situation (towards a new problem), etc.

Our approach is based on mathematical thinking and learning theory of Schwank (1986, 1993) about individual preferences in mental structure, namely predicative versus functional cognitive structures. The teaching process has to reflect these results.

The textbooks and results of researches under guidance Cohors-Fresenborg (Cohors-Fresenborg 1993, Cohors-Fresenborg et. al. 1991 a, b) are a good proof of success of this theory and an important contribution to it. These are very significant for understanding the process of mathematical concept formation. Main ideas and results have formed the foundation of one of our special courses and attracted students attention.

It goes without saying that theoretical framework for our research was constructivist understanding of teaching and learning. We tried to orient teacher students' main efforts towards researching problems connected with the moulding (formation) of cognitive activity methods on the basis of already acquired skills and methods. Though we know that this process is closely connected both with theoretical system of knowledge and with creative thinking.

The first component of students' approach is based on a theory of visual thinking and its development (Arnchame 1981, Luria 1981). The most important stage of visual thinking is a stage of mental framing hypothesis about possible ways of solving problems with analyzing and forecasting their possible results.

The second component is a theory of mathematical creativity development (Krutetzkii 1968) and paradigm of the open-ended approach in mathematics teaching (Nohda 1991, Pehkonen 1995) for the development general competency of scientific style of activity (abstraction, generalization, specialization etc.).

3. The Experiments and Method

Altogether about 70 students were involved in our experiments during 2 years. Every year it was one group of fourth year students and one group of fifth year students. It is natural that during the second year of experiment it was the experimental group that was chosen (the one that had already taken part in the experiment a year ago).

Our hypothesis was that the special courses offered form teacher students' creative features, contribute to successful fulfillment of their own pedagogical researches on cognitive activity. Besides, we thought that using microworld Dynamic Mazes allow to show different cognitive ways of thinking and define which students have difficulties for predicative way of thinking or for functional one. We thought that study of these ideas would greatly increase the effectiveness of teacher students' researches.

Every special course consisted of 30 hours during the spring academic term. During the first year the following special courses were offered:

1. Main ideas and results of curriculum project «Integrating Algorithmic and Axiomatic Ways of Thinking in Mathematics Lessons for grade 7 and 8» using microworld Dynamic Mazes (for the students experimental group of 4th course).
2. Computer Explorations in Math Courses (Plane Geometry, Algebra and Math Analysis) (for the students of 5th course).

During the second year the following special courses were offered:

1. Using Derive system for Solving Problems with Economic Contents (first part) and for Trigonometric Equation Solving (second part) (for the students of 4th course).
2. Computer Explorations in Plane Geometry (first part) Solving Problems with Economic Contents Using Derive system (second part) (for the students experimental group of 5th course).

Thus, one experimental group during two years was given two special courses first of which was connected with the study on research of individual differences in students methods of mental analysis using microworld Dynamic Mazes.

Students performed their course and diploma papers in a constant cooperation and collaboration with their research advisor. Students chose their themes independently out of the proposed list, they also chose the complexity and depth of their researches during its process.

The process of teaching and learning mathematics is divided into three range levels (though all of them are connected the first one is more complicated):

- development of creativity as research thinking;

- acquisition of cognitive activity methods;
- acquisition of pure mathematical knowledge and skills.

According to the above mentioned, we have distinguished three range down levels of students' researches on each of them pedagogical problems imminent to each type of activities were solved: 1) the creativity of non-standard algorithms, independent making of problems, generalization of facts, phenomena, laws, strategies; 2) the activities based on an already known algorithm but with new contents; 3) the reproductive activity (reproducing according to the pattern). The successful solution of these problems was connected with the development of respective intellectual skills.

As our research has shown, the students from the experimental group moved to the first level quicker and easier than others. Besides their diploma papers were characterized by original and independent style, though themes chosen were complex enough. For example: formation pupils' skills and habits of solving trigonometric equations or solving problems of real life, computer explorations in studying surfaces of second order or plane curves.

Besides, the majority of the experimental group mastered the second level. In two other groups there were no results of the kind (though these two groups were different as far as their marks were concerned).

3.1 Some Remarks About Essence of Students' Researches

Computer explorations are very important for the process of studying (cognitive activity) as well as computing experiment for the process of scientific cognition. In the first case, students study subjectively unknown facts, in the second - scientists investigate objectively unknown facts of nature. Therefore teacher students try to develop pupils' exploring ability and to guide (to help) in organizing this process according to some steps: posing out problems; making students' explorations based on computing experiments; framing hypothesis about the way of solving; proving the hypothesis or creating the counterexample to it.

Computer explorations become more effective by Derive and Cabri-geometre facilitating all these steps (graphing and geometric transformations, accurate counting and equations solving, simplifying which produce equivalent expressions and choosing direction of simplifying, etc.). But students have to know many new things, for example: how to prove theorems of well-based hypothesis that close connected with constructing counterexamples (which are capable to refuse wrong hypothesis) or how to choose a compromise between calculation speed and results accuracy, common methods and solution errors. Besides for forming students ability of posing out problems a fruitful approach is a system of problems arranged in series so that the experience of solving the previous problem helps to pose the next one. We suggest (Oleinik et. al.1996) organizing the explorations in four levels for development of exploring pupils' competencies, which provides a success in development of mathematical creativity.

In plane geometry we propose some series of arranging problems which help in posing out problem (and forming hypothesis), for example:

- theorems on the equality (or congruence conditions) of the triangles and theorems on the similarity of the triangles (as for any triangles as for right triangles), properties of sides and angles in triangle;
- theorems on the equality (or congruence conditions) of the angles formed by two parallel lines and intersecting line (internal and external crosswise lying angles, internal and external corresponding angles etc.), theorems on the sum of the angles of triangle; value of external angle of triangle;
- area of right triangle, area of any triangle, Pythagorean theorem, properties of perpendicular, inclined line and projection.

For the acquisition of methods for solving real world problems (applied mathematics) was chosen economical application because many young people are very interested in market economy (ability to solve economical problems is very important in contemporary life). This is the main reason why we have used these topics in the beginning of mathematical analysis teaching and learning. We hope that this approach will provide more than ordinary efficiency of understanding mathematics. Besides for Derive allow rearranging the work time in learning, in particular, we create time to teach new topic "modeling-translating-interpreting".

Computer explorations can be successfully used in solving real life problems as well. Therefore it is important for forming suitable students ability in applied mathematics courses that requires four steps, as usual: choosing a suitable model; translating the real world problem into the mathematical problem; calculating the model solution by applying known methods; interpreting the model solution into a real world solution. For teacher students' researches in this direction we propose to use the method of constructing and reconstructing of problems, to compare the obtaining results from different ways of problem solving, to generalize solving method of this type of problems etc.

For example one economical problem which use equations of second order curves and two reconstructing questions (open-approach method): «1. Investigate (find) the solution of the previous problem when the price of 1 unit of production of the enterprises A and B are change (for example, to accordingly 200 and 225 units of money). 2. Investigate (find) the solution of the previous problem when the transport cost for 1 km is change (for example, from the enterprise A is 2 times less than B).»

The research on possibilities of a system DERIVE concerning the management of trigonometric transformations has shown that they are unique for teaching traditional school methods of their solution (factorization, reduction to a quadratic equation of one functions or homogeneous etc.) as well as more complicated methods.

A teacher's important task is to teach his pupil to see the thing that is instilled in images, i.e. to analyze visual information. It is the discovery of certain fragments and the identification of similar ones (either in form on meaning) that take place first. But the working out of problem solving plan is the most important stage. Derive is very useful in this process as well as in generalization methods of equation solving (discovery of algorithms).

For teacher students' research of in this direction we propose to reconstruct a pupil's possible way of thinking (and to form a pupil's visual thinking) while analyzing the following problem: «To solve the equation $3\sin^2 x - 2\sin^2 x + 5\cos^2 x = 2$.

1. The equation includes trigonometric functions that are why it is a trigonometric equation.
2. The equation includes different trigonometric functions with different arguments.

3. All the summands of the both sides can be represented as a function of one argument.
4. All the summands have a similar degree and we can divide by $\cos^2 x$ for obtaining an equation of one trigonometric function.
5. We know two types of equations simplest $f(x)=a$ and more complicated $af^2(x) + bf(x) + c = 0$.
6. We obtain quadric equation of one trigonometric function.»

The next problem of research is geometric applications of complex numbers (CN) unfortunately which was forgot in a school curricular. Though the method CN allows solving plane geometric problems by the elementary calculations (using the known formulas) which immediately follow from the problem condition. Therefore students' interest to a method CN with using DERIVE is connected to the greater simplicity of its application in a comparison with traditional coordinate and vector methods, which demand considerable quickness of pupils. Even few simplest statements (that are follow from the geometric interpretation of complex numbers) allow to solve rather useful problems on the proof of properties of triangles and tetragons. Besides they allow to prove the known classical theorems of elementary geometry.

4. Some Preliminary Results

The experience realizations these special courses shows, that the use DERIVE and Cabri-geometre support in teaching algorithmic and half-algorithmic problems as well as heuristic problems and therefore heuristic methods of operations that intrinsic of creative activities (abstraction, generalization, specialization etc.).

The evaluation of experiment data allows us to formulate the following results.

1. Students' good knowledge does not guarantee their successful creative activity.
2. Technologies like microworlds and computer algebra system contribute to growing teacher students' interest in psychological researches on cognitive activity.

3. Studying modern teaching methods (for example, the levels and methods of guiding explorations, heuristic methods etc.), the students develop new problems by themselves and investigate new methodical ideas with using technology.
4. Studying essence of the theory of individual preferences in mental structure by computer microworld increases the effectiveness of students' researches and shows great students' interest to the development of practical recommendations of this theory.
5. The considered approaches provide a success in teacher students' researches on cognitive activity ("open-end" problems in algebra and geometry; complex numbers and their geometric applications; elements of linear programming; moving and rotation objects on the plane (tangram) etc.)

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ON PREDICATIVE VERSUS FUNCTIONAL COGNITIVE STRUCTURES

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Abstract: *Predicative thinking is thinking in terms of relations and judgments; functional thinking is thinking in terms of available actions and achievable effects. Depending on the way of thinking the orientation in the world, the type of sources for getting insight are not the same. E.g. it should be visible in different eye movements. In addition to our qualitative experiments, recently we started to run a study based on EEG-methods while students were solving logical pattern fitting tasks. The EEG complexity during predicative thinking decreased in comparison to functional thinking and mental relaxation, with this reduction being most pronounced over the parietal and right cortex. A reduction in dimensional complexity during functional thinking which was concentrated over the left central cortex, although significant, was less clear.*
Keywords*: cognition, EEG, eye movement.*

1. Introduction

Concerning cognitive models as sources for intelligent behaviour like planning or learning different approaches have been studied during the years. There is broad agreement - since the time of Plato - that language and - as a newer approach in our century - visualization is a tool of thinking. Together with the cognitive turn as movement against behaviourism lots of statements indicating that thinking equals language were set.

There are good reasons to accept another cognitive base of elaborated thinking, which is related in a specific sense to motoric-guided cognitive representations. At an interdisciplinary conference on “Thinking and Speaking” the mathematician van der Waerden (1954) pointed out, that thinking in motoric terms is on top level in creative mathematical work: it might be that someone is a visualizer and therefore has more benefit in thinking in images, but for sure the words which are used to label the mathematical concepts only play a minor role. This seems to be in contrast e.g. to what is used of the Piagetian theory of levels of cognitive development, where motoric representations are on the bottom of the hierarchy and it is the formal representations which are on the top. In our theory we use the term *functional thinking* for motoric thinking in the sense of van der Waerden (which includes only such motoric actions useful for productions) and we contrast it with a kind of thinking that doesn't care so much on dynamics but on static structures and the embedded complex relationships, this latter we call *predicative thinking*. Up to now there is not much clearness about the relationship of functional/predicative thinking and all kinds of visualization and imagery. It seems to be clear that in case of predicative thinking good language knowledge is useful, because it is of much advantage in creating an adequate structure in a problem solving situation to use word-labels to build up this specific structure.

Beyond the fact that the functional way of thinking plays an important role in mathematical thinking its range of appliance is even much broader. Bateson (1980[4], 120-121) presented a nice example concerning the difference we are interested in:

»We do not notice that the concept "switch" is of a quite different order from the concepts "stone," "table," and the like. Closer examination shows, that the switch, considered as a part of an electric circuit, *does not exist* when it is in the on position. From the point of view of the circuit, it is not different from the conducting wire which leads to it and the wire which leads away from it. It is merely "more conductor." Conversely, but similarly, when the switch is off, it does not exist from the point of view of the circuit. It is nothing, a gap between two conductors which, themselves exist only as conductors when the switch is on. In other words, the switch is *not* except at the moment of its change of setting, and the concept "switch" has thus a special relation to time. It is related to the notion "change" rather to the notion "object".«

In some studies (e.g. Schwank 1993a, Schwank 1994) on cognitive ways of learning basic concepts of computer science we used within a set of bricks (Dynamic Mazes, www.ikm.uos.de/aktivitaeten/dynamische_labyrinthe.htm) a mechanical switch (Fig. 1) for representing a tool for case distinction. For instance, by means of this switch organisational problems like automatical bottle-selling procedures can be solved. The key point is to plan actions and thereby to anticipate their influence on later occasions, which is a typical functional requirement. Predicted difficulties of part of the students based on their individual problems dealing with functional concepts arose (e.g. Schwank 1994).

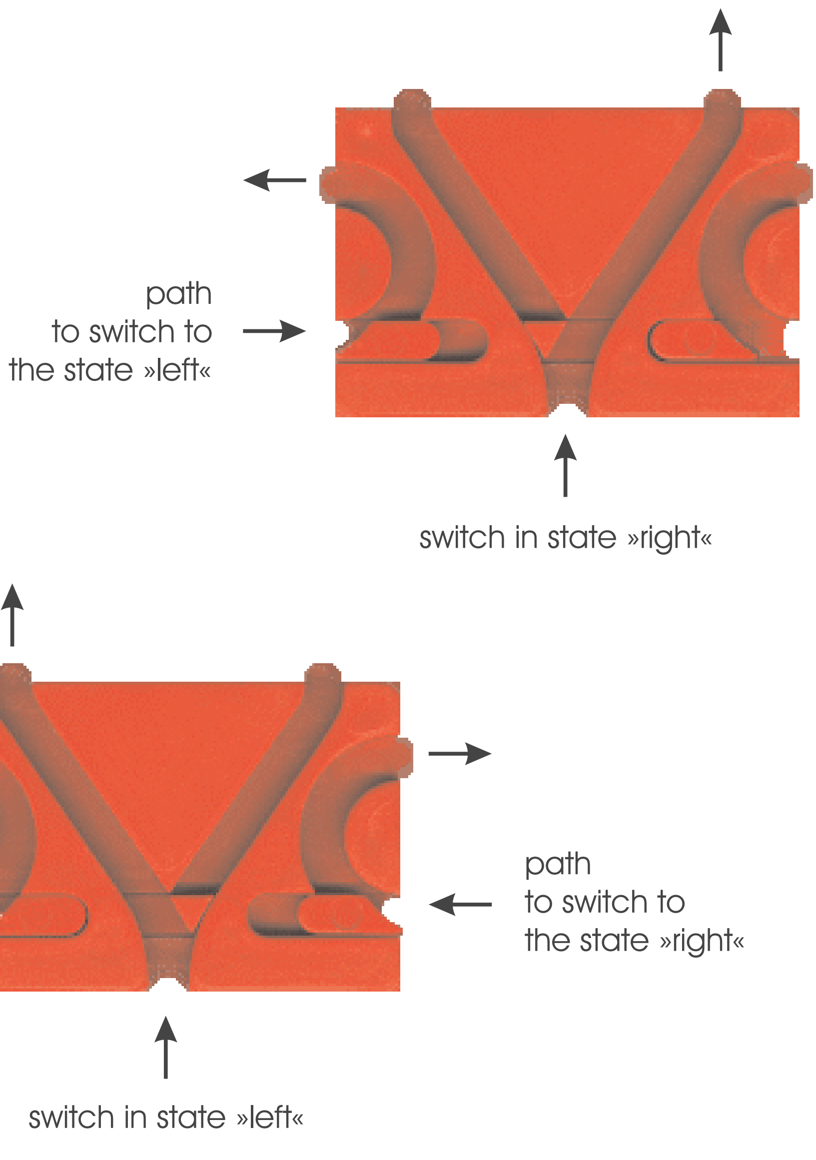

Fig. 1: Switch of the Dynamic Mazes
(Test it.)

In the following we first give a short overview of our theory and than present some examples of short logical tasks, which we are using at present to check abilities and preferences of subjects in using a functional or a predicative cognitive structure.

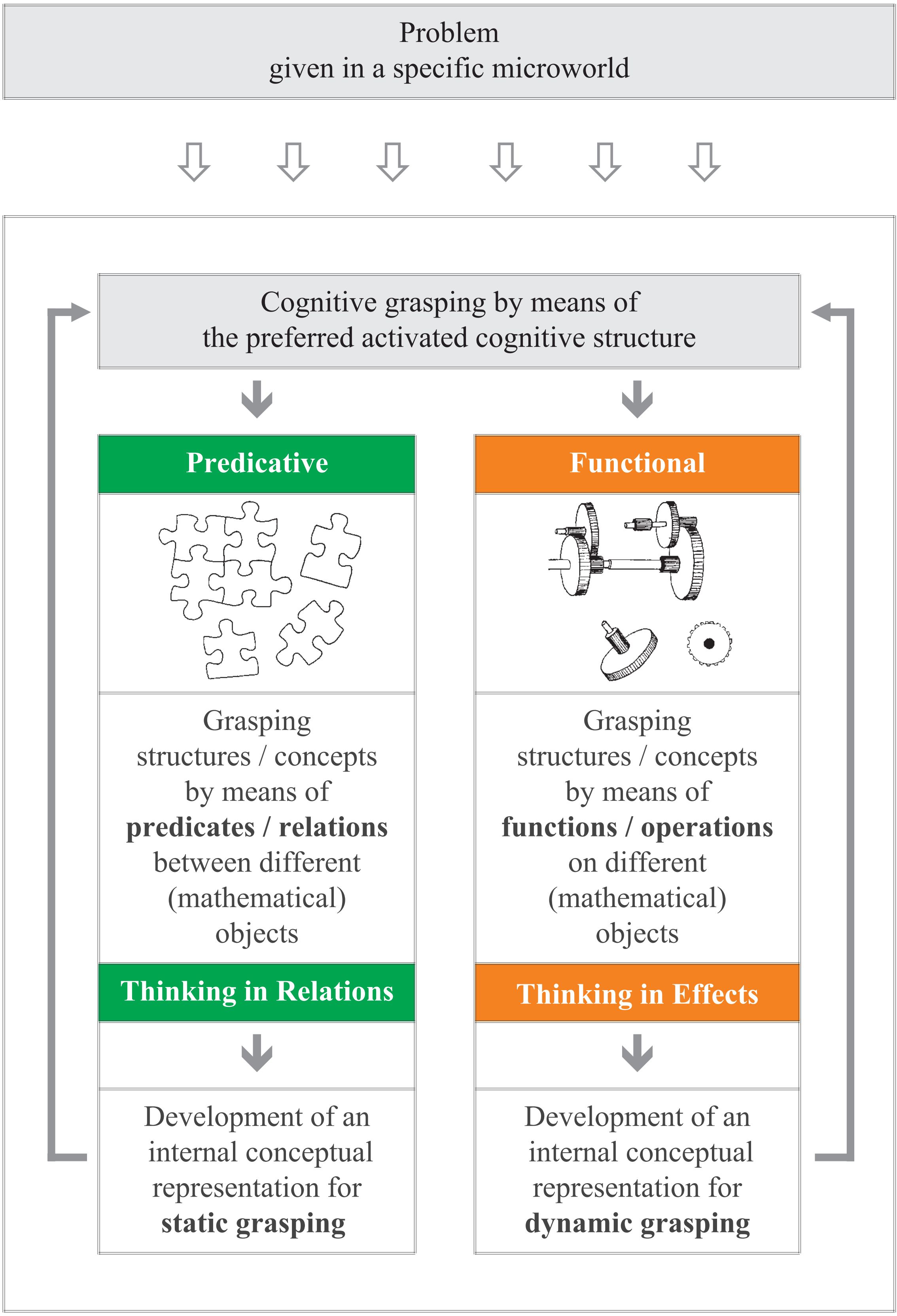

Fig. 2: Predicative versus functional cognitive organisation (cf. Schwank 1995)

2. Predicative versus Functional Cognitive Structures

We distinguish between static and dynamic mental modelling as a characteristic of the individual cognitive structure in terms of predicative versus functional thinking (Fig. 2). Predicative thinking emphasizes the preference of thinking in terms of relations and judgments; functional thinking emphasizes the preference of thinking in terms of courses/effects and modes of action (cf. Schwank 1993a, 1996). For an overview of the experimental testing of the theory see Schwank (1995, 108-115). Research has also shown that it is quite rare to find female subjects who behave in a functional way (see also Schwank, 1994) or that functional and predicative thinking occurs as well in Indonesia (Marpaung 1986) and China (Xu 1994).

The given diagram (Fig. 2) has to be read spirally in chronological order. The black arrows describe circles in order to consider that the internal tools of the conceptual representation influence that which will be grasped cognitively. In consequence the further development of the internal conceptual representations is interfered. The observed differences in behaviour partly are explained in such a way that both kinds of cognitive structures are not applied equally which results in a different development of a more static or a more dynamic internal conceptual representation.

The category of individual cognitive structures has to be separated from the category of individual cognitive strategies. We distinguish between a conceptual, top-down organising, and a sequential, more interactive approach (Cohors-Fresenborg & Schwank, 1996). Predicative / functional refers to the tools of thinking, conceptual / sequential refers to the global organisation of the problem-solving process. Concerning links to other cognitive theories such as declarative/procedural see e.g. Schwank (1993b).

3. Examples

For a predicatively structured person the central point of his or her analysis concerning a complex situation is to break it down into different conceptual pieces and to invent a logical structure which describes the network of the relations between these pieces. For a functionally structured person the central point is to arrange the going through the production as a complex process in which different strengths control, determine or promote each other. For the former the mental model describes the logical structure, for the latter it describes the organisation of work flow in time.

To show the benefit of our cognitive theory for e. g. a cognitive approach in business reengineering (Cohors-Fresenborg & Schwank 1997) or a cognitive science approach in computer programming (Schwank 1993 a,b) we have designed different studies which are run with single subjects using different settings: fitting figures in matrices (QuaDiPF), organising processes in a microworld (OPM). QuaDiPF (Schwank, 1998) is a qualitative diagnostic-instrument to determine the preferred cognitive structure, predicative versus functional. In OPM those tested have to solve a sequence of organisational problems with the specific microworld Dynamic Mazes (cf. Cohors-Fresenborg, 1978, www.ikm.uos.de/aktivitaeten/dynamische_labyrinthe.htm). This is the mechanical realisation of a mathematical idea of automata which is equivalent to the Turingmachine. We know from our studies that this setting in the beginning supports the functionally structured subjects. While solving the more complex problems a predicative cognitive structure is more successful. Here we concentrate on QuaDiPF because the setting is much simpler and not as time consuming as OPM. Finally this newer analysing tool is even usable in EEG-measurement-environments.

3.1 Fitting Figures in Matrices: QuaDiPF

We use tasks such as those in common intelligence tests (e. g. Raven, 1965) to find a missing figure, which fits suitably into a set of 8 given figures arranged in a matrix. In a clinical interview each subject has to invent and draw the missing figure in the matrix (instead of selecting it from a given set as usual). The subject has to argue why he or she drew this very figure. The analysis of the videotapes shows that a predicative and a functional way of mentally modelling the task exist. In a predicative mental model the subject uses *predicative tools*, e.g. looking for properties, inventing general laws. So, in the given example (Fig. 3a) the subject tries to structure the image. Each figure consists of three objects: a star, a point and a circle. The triangle is the same in each figure. In each row the point is at the same place. In each column the circle is at the same place. In

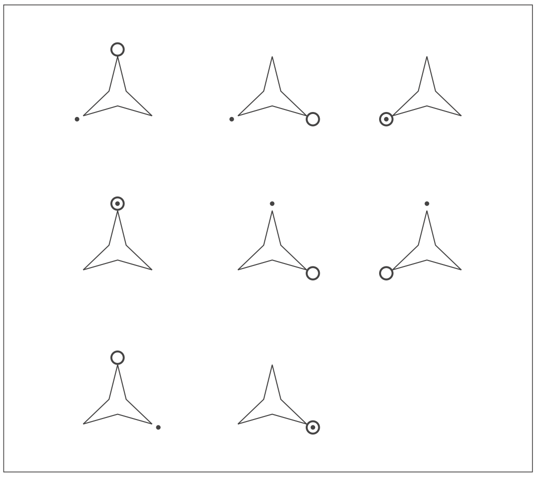

Fig. 3a: QaDiPF-Example (Schwank, 1998)

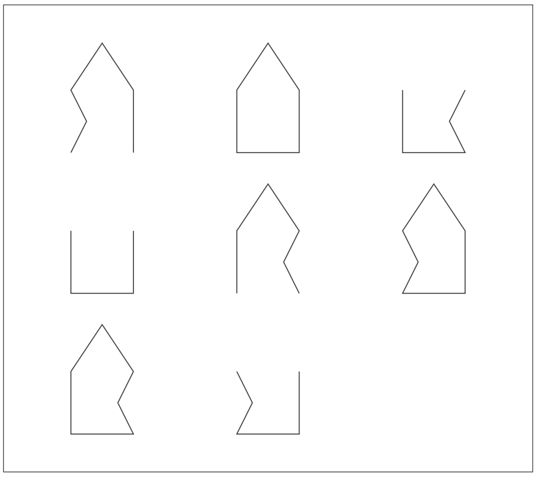

Fig. 3b: QaDiPF-Example (Schwank, 1998)

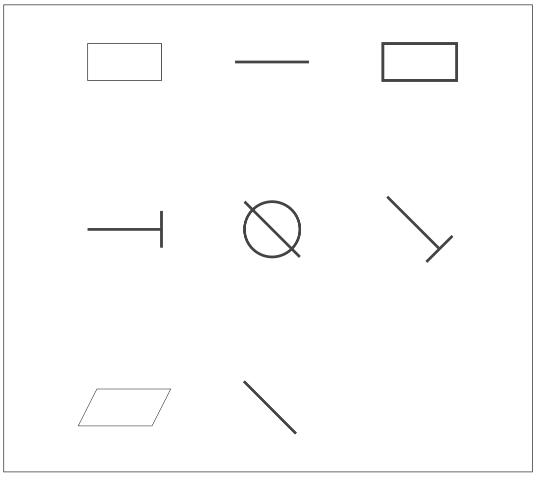

Fig. 3c: QaDiPF-Example (Schwank, 1998)

a functional mental model the subject uses *functional tools*, e.g. invents a process which produces the last element in a row or column. In each row the circle moves around, and in each column the point moves around. The object around which the movement takes place does not change. In both ways of dealing with the problem the result is the identical.

Besides tasks such as 3a we also invented tasks which are either easier using a predicative analysis or a functional one, so that in the end we established a new type of intelligence test. Fig. 3b shows an example in which a predicative analysis is useful to construct a working mental model. The main idea is to invent a structure by arranging the properties. One could, for example, proceed as follows: three types of figures exist (closed figures, figures which are open at the top and figures which are open at the bottom) which each have straight walls, bent left walls and bent right walls. The figure with an open bottom and straight walls is missing (composition of predicates).

Fig. 3c shows an example in which functional analysis is useful. The main idea is that the figures in the middle row and the middle column symbolise operators. One could, for example, proceed as follows: in the first row the first figure is given thick lines by means of the operator. In the first column the first figure is pushed by the operator and transformed into a parallelogram. In the second row the first figure has to be turned by means of the operator. In the last line as a consequence the first figure has to be turned and it has to be given thick lines (concatenation of operators).

We have designed the tasks in QuaDiPF in the form that the subjects have to explicitly construct the missing figure instead of selecting it from a given set of possibilities, for the following reasons: we are interested in the nature of thinking processes and the omission of possible solutions makes the tasks more difficult. Furthermore, we are interested in the individuality of problem-solving: a given set of possible solutions could influence the way in which the tasks are analysed. As a consequence our methodology is rather a qualitative one than a quantitative one.

In the literature it is discussed that solving this kind of tasks requires especially inductive thinking (e. g. Klauer 1996). Our findings show that not only one kind of inductive thinking exists. In a predicative model induction means abstraction. The result is a predicate which is fulfilled by the given examples. In a functional model induction means generalisation. The result is a function which produces the given examples (cf. Cohors-Fresenborg & Schwank, 1996).

4. EEG-Study

Fig. 4: Phase in the EEG-Study

Together with Jan Born and his group, Medical University of Luebeck, we run a study "Dimensional complexity and power spectral measures of the EEG during functional versus predicative problem solving" (Mölle et al., in print). The EEG was recorded in 22 young men (Fig. 4; students at the Medical University of Luebeck) while solving QuaDiPf tasks.

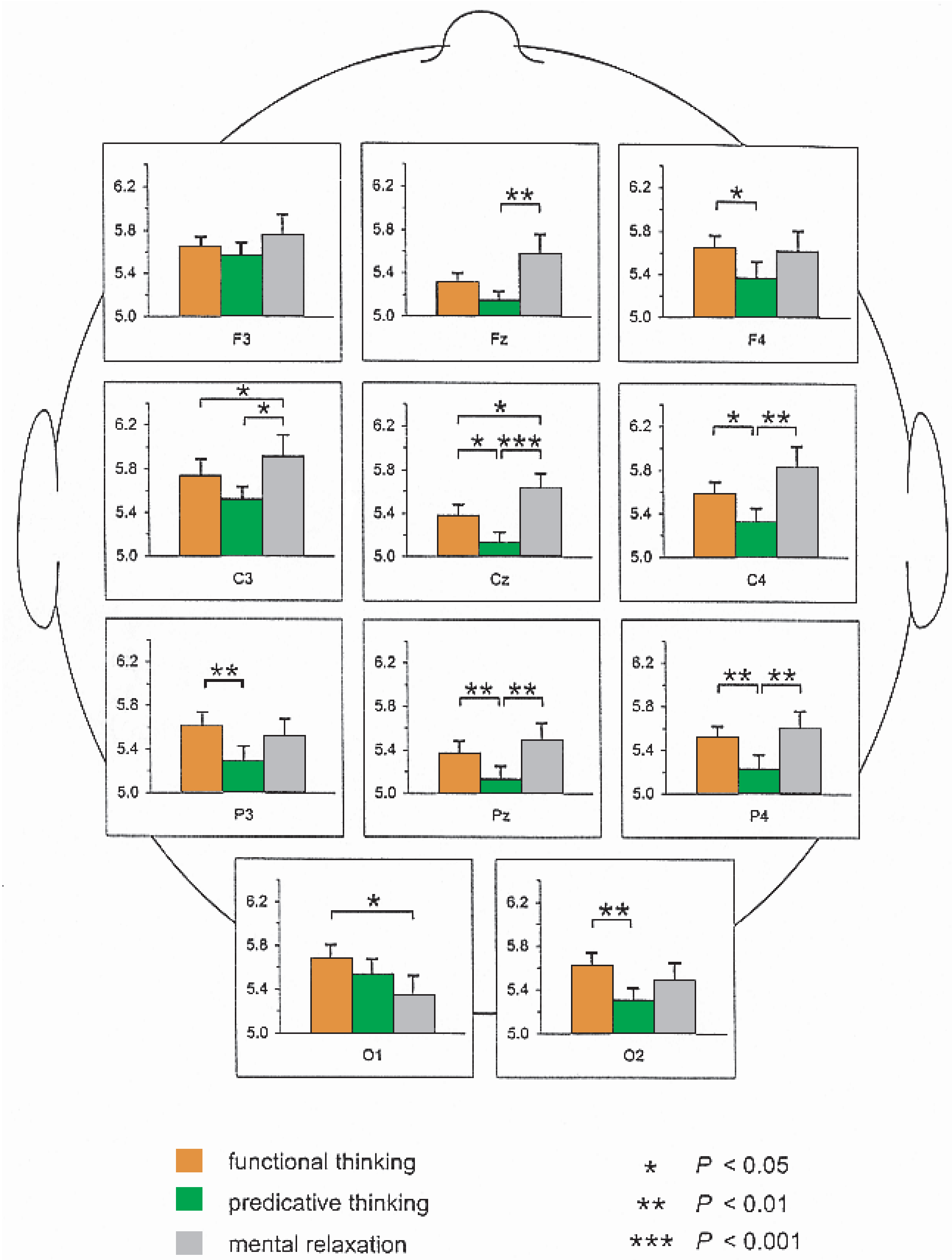

Fig. 5: Results of the EEG-Study (cf. Mölle et al., in print)

Because of known gender differences in brain activities, it was important to work with subjects of equal sex. We decided to start with male subjects, hoping that among them will be enough typical thinkers in both modes. It turned out, that we were lucky in the predicative case, but not the like in the functional case. Familiar with the behaviour of the students in our department of mathematics and computer science in Osnabrueck, the Osnabrueck staff was quite a bit astonished, that the young medical men didn't show up so much an ability to good functional reflections. Further experiments have to follow. In this first experiment the subjects performed on three different blocks each including 4 QuaDiPF tasks. After having completed a pattern mentally by his own the subject had to draw his solution and to explain why it fits the pattern well (Situation in Fig. 4). In the first block the subjects were asked to solve the tasks spontaneously. In the second and third block the subjects were primed to do it in a functional or a predicative way respectively. The priming took place during three tasks, for that purpose a typical functional or predicative argumentation for the solution was presented. The subject was asked to solve the fourth task just in the way it had been shown to him. The EEG during thinking on the fourth task of each condition was taken for analysis. The EEG complexity during predicative thinking decreased in comparison to functional thinking and mental relaxation, with this reduction being most pronounced over the right and parietal cortex. A reduction in dimensional complexity during functional thinking as compared to mental relaxation which was concentrated over the left central cortex, although significant, was less clear (Fig. 5).

5. Eye Movement - Reflections

Especially tasks like 3c put some problems in adequate handling. Carpenter et al (1990) tried to analyse the manner how subjects solve the Raven Matrices using eye-tracking methods. They rely on a (predicative) classification of the Raven tasks, which worked except for one task (APM No. 18 / isomorphic to fig. 3c): "*Problem was not classifiable by our taxonomy*" (Carpenter et al. 1990, p. 431). This very task No. 18 is in its style unique in the APM-Test. By means of an added functional classification on one hand such "mysterious" tasks could be approached systematically as well and on the other hand concerning tasks with a "double" nature like fig. 3a there could be offered, additionally to the predicative classification, a functional classification. Therewith a broader understanding of the subjects cognitive behaviour is possible.

In the near future we will run eye-tracker experiments using QuaDiPF tasks. We are convinced, that the results will support the theory of functional/predicative thinking. We expect, that we will found different patterns of eye-movements which are either specific for a predicative analysis or a functional analysis: Like in Carpenter et al. (1990) we should find eye-movements following essential properties (Fig. 6a), but moreover we should find eye-movements along the production process of the step by step developing states of specific objects (Fig. 6b).

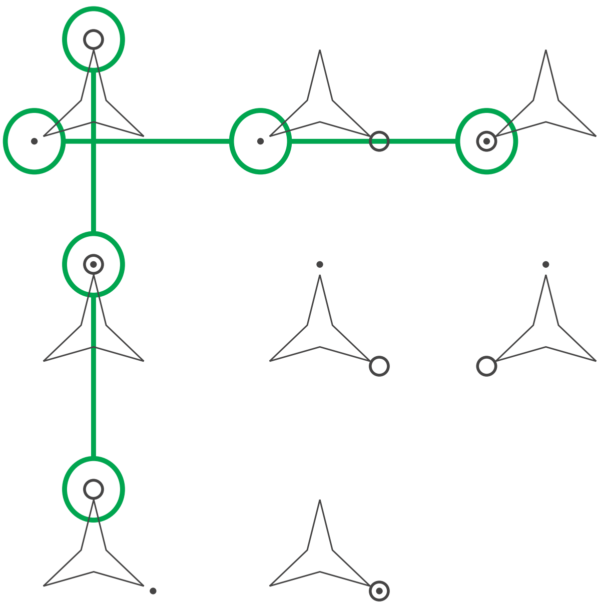

Fig. 6a: Eye movement during predicative analysis

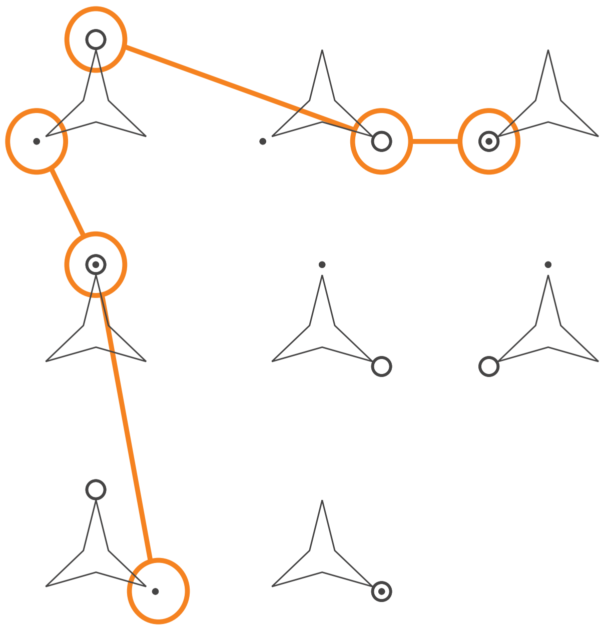

Fig. 6b: Eye movement during functional analysis

6. Outlook

When we started to run our research work we could show that the distinction between functional versus predicative thinking is useful to analyse the behaviour of subjects in problem solving situations in the field of mathematics and computer programming. Viable concepts (in the sense of von Glasersfeld 1995) there can be created in one or the other manner. Later on we could even focus on the nature of these two thinking modes using much simpler tasks of a nonverbal intelligence test, which turned out to be usable in EEG-measurement-environments.

In our days we still see the problem that it is much easier to communicate predicative ideas than functional ones. We agree with Vandamme, that the problem will be to find more adequate representations for functional ideas (Vandamme calls them: action oriented, which we don't like so much because there do exist predicative actions, e.g. in solving a jigsaw puzzles and we want to stress the van der Waerden aspect of motoric thinking in terms of construction instructions). We are convinced, that the new technologies which enable to create virtual and augmented reality (e.g. Vandamme & Morel 1996) and which bring up new tools to easily create and manage dynamic actions on the computer screen are actually the appropriate means to master this challenge.

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WINNING BELIEFS IN MATHEMATICAL PROBLEM SOLVING

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Abstract: *The aim of this study is to define the relationship between the ability in solving mathematical problems and the beliefs about mathematical problem solving. We compare the beliefs of children with or without difficulties in solving mathematical problems, by using a questionnaire. The results show that good solvers and poor solvers have significantly different concepts of a mathematical problem: some beliefs appear to be winning in that they are able to activate the correct utilization of knowledge.*

Keywords: *beliefs, problem-solving, school problems.*

1. Introduction

Several studies in the field of the teaching of mathematics are concerned with the emotional-motivational component of learning (Schoenfeld 1983; Lester 1987; Mc Leod 1992). It has been shown that the failure in solving problems is not only due to the lack of knowledge, but also to the incorrect use of knowledge which is often inhibited by both general and specific beliefs about mathematics.

Beliefs consist of the subjective knowledge that an individual develops in attempt to interpret the surrounding environment (Lester 1987): they influence how the subject learns since they represent the context in which the subject selects and uses cognitive abilities (Schoenfeld 1983; Masi, Poli, & Calcagno 1994).

Some common beliefs of general type associated with Mathematics are (Schoenfeld 1985):

- Mathematics problems are always solved in less than 10 minutes, if they are solved at all.
- Only geniuses are capable of discovering or creating mathematics.

Specific beliefs about mathematics are:

- Since 31>5, then 0.31>0.5.
- The number -a is always a negative number.

Besides the beliefs associated with Mathematics, the beliefs about self are important as well.

Borkowsky and Muthukrishna (1992) have suggested a model that relates behavioral patterns of children facing school tasks, metacognition abilities, and self. In their model, motivation, system of the self, development of correct learning strategies, and self-regulation processes are closely linked. They defined a child with the following features as a good information processor: high self-esteem, internal locus of control (Weiner 1974), causal attribution of success and failure to effort, incremental theory of intelligence (Dweck & Leggett 1988), and feelings of self-efficacy.

Our interest was focused on the emotional and motivational components of this model, in particular the implicit and explicit beliefs about self - self-esteem, role of effort, attributional theories - and learning (both the object of learning and school environment).

More specifically, the aim of our study was to define the relationship between solving mathematical problems, and the beliefs about self and mathematics.

To realize this we have elaborated a questionnaire (the Beliefs about Mathematical Problems Questionnaire) with open and multiple choice questions: in the latter case the proposed options were constructed on the basis of a previous research (Zan 1991 and 1992), that aimed at identifying, through open questions, the conceptual model of real life problems and of school problems possessed by primary school pupils. In the course of this research one of the following three questions was proposed to 750 primary school pupils: “What is a problem for you?”, “Give an example of a problem.”, and “What comes to your mind when you hear the word *problem*?”

Then we compared the answers to BMPQ given by children with or without difficulties in solving mathematical problems. The findings presented in this paper will concern the beliefs about mathematics, in particular the concept of a mathematical problem.

2. Method

101 children aged from 8.5 to 10.9 participated in this study. The children were in three, four and five grades in the primary schools in Pisa and Livorno, Italy.

The Mathematical Problem Solving Task, characterized by four standard mathematical problems, was presented collectively in order to select two groups of children. Children were classified in two groups: a group of *good solvers* composed of 30 children who solved all the mathematical problems, and a group of *poor solvers* of 21 children who solved one or none of the mathematical problems. Therefore, the final sample was of 51 children.

We administered the Beliefs about Mathematical Problems Questionnaire to investigate the concept of a mathematical problem. Questions were read aloud by the examiner and there was no time limit to answer.

3. Results

Owing to lack of space, we discuss in detail only the answers to the most significant questions. In particular, we omit the answers to the open questions and the justifications for the multiple choice questions, as they would require a more complex analysis.

For simplicity, we have inserted after each question the associated table with the relative data and comments. In order to decide whether the difference between the two groups (good solvers / poor solvers) is significant, we have collected together the answers to some of the questions, naturally on the basis of theoretical criteria.

3. In your opinion, why are mathematical problems called *problems*?
(Choose just an answer)

[A] This is a usual word to call them: they might also be called “exercises”.

[B] Because for the mind there is a difficult situation to solve.

[C] Because for a child who is unable to solve it, it becomes a problem.

[D] Because they describe someone’s problem and we are asked to solve it.

Question 3 aimed at deciding whether the children reduce the concept of mathematical problem to the more general concept of real problem: this is possible in several ways (answers B, C, D). In case C, unlike B and D, the problematic situation is external to the task and centered on the subject. From a different point of view, the children who choose B recognize as problems a broader variety of situations in comparison with those who choose D. These remarks are summarized in the following table:

Question 3	**Reduction Sch.pr. - Real pr.**		
A	No	**Involvement**	
C	Yes	ego-centered	**Generality**
D	Yes	task-centered	low
B	Yes	task-centered	high

In our opinion, the answers A and C correspond to less effective approaches. Because of their small number, they are collected in one group.

Most good solvers choose answer B, whereas most poor solvers choose answer A or C. [chi^2=6.153; p=0.047]

Question 3	Good solvers	Poor solvers
A or C	6	11
B	17	6
D	7	4

4. What is a mathematical problem?

[A] It is a text with some numbers and a question.

[B] It is a situation that you can solve by using mathematics.

[C] It is an exercise where one has to decide which operations should be done and then do them.

[D] It is an exercise presented during a mathematics lesson at school.

Question 4	Good solvers	Poor solvers
[A]	4	4
[B]	23	5
[C]	2	6
[D]	1	6

Most good solvers choose answer B, whereas most poor solvers choose answer A or C or D.
[chi^2=13.939; $p<0.001$]

5. Does there exist a mathematical problem without numbers?

Question 5	Good solvers	Poor solvers
Yes	21	8
No	9	13

[chi^2= 5.126; p=0.024]

9. Alessandro says: "A problem with many questions is more difficult than a problem with one question." Do you agree with him?

Question 9	Good solvers	Poor solvers
Yes	6	11
No	24	10

[chi^2= 5.828; p=0,018]

10. Alice says: “A problem with a short text is easier than one with a long text.” Do you agree with her?

Question 10	Good solvers	Poor solvers
Yes	3	13
No	27	8

[chi^2=13.8 with Yates correction; p< 0.001]

28. In a problem is it worse to make a calculation error or to choose the wrong operations?

[A] Calculation error.

[B] Choose the wrong operations.

[C] It’s the same, there is no difference.

Question 28	Good solvers	Poor solvers
[A]	8	15
[B]	20	5
[C]	2	1

[Considering only answers A and B, chi^2=8.303; p=0.004]

34. How do you feel when the teacher says: “Now let’s do a problem.”?

[A] You are excited but happy.

[B] No particular feeling.

[C] You are nervous because you don’t know if you will be able to solve it.

[D] You are quite scared.

Question 34	Good solvers	Poor solvers
[A]	14	2
[B]	11	1
[C]	1	5
[D]	4	13

We have collected together the answers corresponding to positive or neutral emotions (A and B), and those corresponding to negative emotions © and D).
[chi^2=21.649 with Yates correction; $p<0.001$]

4. Conclusions

We distinguished four categories of children based on the analysis of the responses given to question 4 of the questionnaire:

1. Formalists - they recognized the mathematical problem on the basis only of the formal features of the text [i.e., it is a text with some numbers and a question];
2. Structuralists - the mathematical problem was identified by using mathematical tools [i.e., it is a situation that you can solve by using mathematics];
3. Operatives - the presence of arithmetic operations defined the mathematical problem [i.e., one has to plan the arithmetic operation and do it];
4. Pragmatist - for these, the mathematical problem was characterized by contextual elements [i.e., it is presented during a mathematics lesson at school].

The good solvers significantly belonged to the category of structuralists.

Furthermore:

- The motivation of good solvers was focused on the task; on the contrary, the motivation of the poor solvers was self-centered;
- The poor solvers were more sensitive to syntactic cues - text length, number of questions, magnitude of numbers - than the good solvers in judging the difficulty of a mathematical problem;
- The poor solvers considered an error in calculation worse than one in planning or selecting the correct arithmetic operations;
- The poor solvers were aware of being anxious when facing a mathematical problem.

In conclusion the good solvers and the poor solvers have a significantly different concept of a mathematical problem.

The difference between the answers of the two groups suggests the following definition. We call "winning" the beliefs of the good solvers, because these beliefs appear to be able to evoke the correct utilization of knowledge.

Metacognitive research has shown that, through a direct intervention, the beliefs about self can be modified. In this way, the cognitive and metacognitive resources can be activated.

Hence, in view of this research, our results suggest a teaching approach to difficulties in problem solving that makes children's beliefs explicit and endeavours to modify them.

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GROUP 6: SCHOOL ALGEBRA:EPISTEMOLOGICAL AND EDUCATIONAL ISSUES

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SYNTHESIS OF THE ACTIVITIES OF WG-6 AT CERME 1

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1. General Information

Pre-conference preparation was coordinated by C. Bergsten (Sweden), P. Boero (Italy) and J. Gascon (Spain). Ten contribution proposals were received, eight of which were accepted, distributed to all participants before the conference and discussed during the conference. Coordination of Working Group activities was ensured by C. Bergsten and P. Boero. From ten to twelve participants took part in the WG sessions; nine of them were present at all sessions. Countries represented within the WG were France, Germany, Hungary, Italy, Spain, Russia and Sweden.

2. Methodology

Given the content of the papers, it was decided to distribute the five time slots at our disposal as follows:

- three hours for the topic “The nature of algebra related to the teaching of algebra” (discussion mainly concerned the papers by M. Bosch & al and S. Rososhek);
- an hour and a half for “Arithmetic and algebra in schools” (paper by N. Malara);
- three hours for “Tools for research in the teaching and learning of algebra” (papers by C. Bergsten and by E. Szeredi & J. Torok);
- an hour and a half for “Equations” (paper by M. Reggiani);
- three hours for “Construction and interpretation of algebraic expressions related to other domains” (papers by L. Bazzini and by V. Hoyos & al).

The papers discussed in each time slot were intended as starting points for the discussion. In reality, many links were made between the first and the third topics, and between them and the other topics. The discussion of papers usually started with previously prepared reactions from some members of the group. The papers were not presented during the WG sessions; however, during the discussion of a paper, the author was sometimes asked to provide "local" information (examples, explanations, etc.) or to contextualise the content of that paper (especially in the case of pieces of research belonging to wide-ranging projects).

3. Summary of Discussions

The following summary is organised according to the topics listed above; it outlines a few points discussed during the sessions of the working group, in order to give the "flavour" of the debate. It is interesting to note that some contributions on one topic came from discussions related to another topic.

The summary has been checked and improved by the members of the WG.

3.1 The Nature of Algebra Related to the Teaching of Algebra

During our discussions we considered different "definitions" of algebra:

- "absolute definitions", i. e. not specifically related to teaching situations; for instance "study of algebraic structures", "modelling of mathematical activities", "specific activities related to the use of algebraic language", etc.
- definitions concerning "school algebra": for instance, "generalised arithmetic", "mastery of algebraic language", etc.

Each work in mathematics education concerning algebra is based on a specific epistemological model of algebra, even if sometimes the definition used is not explicitly presented or is taken for granted. The particular definition of algebra that is assumed has different implications for the teaching and learning of algebra in schools (aims, content, methods). For instance, the following aims of the introduction of algebra in the curriculum are related to the Weyl-Shafarevich's conception of algebra:

to overcome syntactical difficulties in the mastery of algebraic expressions; to develop students' imagination in mathematics; and to enhance metacognitive processes.

It was noted that it is useful for research in mathematics education to consider different models of what algebra (and learning algebra) is, and to "play the game" within the chosen models. As an example, algebra was considered a process of modelling a whole mathematical work (which includes types of problems, different kinds of techniques to solve these problems, and theoretical tools to "talk about", justify and expand the resolution of these problems). According to this perspective we can say that the introduction of algebraic tools needs a previous construction of a whole mathematical activity (be it numerical or geometrical). With this "anthropological definition of algebra", we can also show some mathematical and didactic restrictions that limitate the introduction of algebraic tools in schools and can explain why secondary school curricula are only partially algebraized. Nevertheless it was remarked that this way of considering algebra is weak on the important symbol transformation aspects of algebra. Another example viewed algebra as a matrix of the dimensions "content" vs "activity", i.e. equations, generalisations, and relations vs translating, transforming, and interpreting. This model is close to school algebra practice, but is weak from the meta level perspective, since it is more describing than analysing the whole setting.

3.2 Arithmetic and Algebra

If we consider "school algebra" not as "generalised arithmetic" but as a systemic activity related to the use of the algebraic language, some activities in the field of elementary arithmetic can be performed in an algebraic perspective ("teaching arithmetic in an algebraic mode"). As an example, we may consider (in the first grade) the activity of representing the number 7 in different ways: 7=4+3=6+1. But this kind of activities has raised an interesting issue: the teacher's conceptions about algebra as a possible source of obstacles in the direction of an arithmetic curriculum partly oriented in an algebraic perspective. As concerns this perspective, teacher's difficulties are not local (i.e. related to lack of some specific notions), they are global. So the aim of early introduction of basic algebra skills in primary school must take account of teachers' pre-service training and conceptions and their possible evolution through suitable in-service training based on reflection about classroom experiences related to the nature of the discipline.

3.3 Tools of Research in Teaching and Learning of Algebra

Different theoretical frameworks and tools belonging to various disciplines related to mathematics education were extensively presented during the discussion by people which currently use them in their research work: two specific theoretical frameworks in the area of mathematics education:

- Brousseau's "Theory of didactical situations";
- Chevallard's "Anthropological theory";

and other frameworks and tools from different disciplines:

- Johnson & Lakoff's "image schematas";
- other tools belonging to cognitive psychology;
- Weyl-Shafarevich's epistemological position about algebra;
- teaching and learning implications of Frege's distinction between "sense" and "denotation" of a sign;
- epistemological obstacle (as an interpretative tool for students' and teachers' behaviours).

Two important and delicate questions were repeatedly tackled: What research problems in the teaching and learning of algebra are related to what theoretical frameworks? And is there a difference between specific (to mathematics education) and non specific (taken from other disciplines) frameworks?

Discussion concerned the "generative" character of theoretical frameworks in the definition of research problems and of teaching and learning phenomena.

Another (meta-)research question concerned the possible need for complementarity of tools belonging to different disciplines and theoretical frameworks in order to investigate specific aspects of the teaching and learning of algebra. For instance, the misuse of the symbol Σ by a student during the solution of a "generalization and proof" problem was interpreted from the anthropological point of view (considering the use of Σ as a part of a mathematical technique that might have been unavailable to the student) and the cognitive point of view (considering the difficulties inherent in the mental processes involved in the use of Σ).

The difficult problem of validation of research tools was also raised.

Finally, the question of the validity of "comparative studies" concerning the teaching and learning of algebra was discussed. If the many relevant variables are taken into account, it seems very unlikely at this moment that valid comparative studies can be done. Nevertheless, comparisons might be very useful: they might suggest interesting research issues, research hypotheses, etc.

3.4 Equations

Two research questions were focussed on:

A) aims (and concrete possibility) of early reflective activities about equations (e.g. discussing the equivalence of 0.x=5 and 0.x=10); must reflective thinking come necessarily after action? Must the concept of equation be extensively practised (by solving standard equations), before reflection be started?

B) what theoretical frameworks (in the field of didactics of mathematics and in the field of cognitive psychology) are needed to tackle the preceding issue? For instance, a curriculum based on the relevance of Frege's distinction between "sense" and "denotation" of a sign tends to bring forward the activities indicated in A. And analysis of the difficulties (met by students in tackling those activities) in terms of the "didactical contract" points out the break between the arithmetic and algebraic approaches to equations. Indeed the algebraic and the arithmetic perspectives imply an important change in the meaning of the same activities (for instance, solving an equation).

3.5 Construction and Interpretation of Algebraic Expressions Related to Other Domains

Some research questions emerging from the discussions were:

- the nature of the difficulties in the transition from verbal expressions (related to arithmetic, geometry, etc.) to algebraic expressions. Different interpretative hypotheses were suggested, some related to semiotics (different organization of verbal and algebraic languages), others to cognitive psychology (based on the different image schematas involved), and others to didactics of mathematics (familiarity with different kinds of techniques; familiarity with this kind of "translation" task);
- potentialities (and possible limitations and ways to overcome them) inherent in the dynamic environment of Cabri software in the transition from geometric construction of curves to their algebraic expressions as equations. The possibility of easy "trials" could induce students to avoid the effort of elaborating suitable algebraic expressions through a deep analysis of the curve. But this danger could be avoided through a suitable choice of tasks related to the specific characteristics of the software.

4. General Comments

The "circumscribed" topic and the sound scientific basis of the accepted papers produced discussions that were very productive in terms of communication and mutual understanding. These two aspects prevailed over the collaboration perspective; perhaps no other outcome was possible given the divergence in the research traditions and frameworks represented. However, some hints for collaboration did emerge: for instance, testing different research tools on the same "corpus" of protocols; and comparing different approaches to algebra (all based on preliminary activities in the domain of arithmetic).

ON THE CONSTRUCTION AND INTERPRETATION OF SYMBOLIC EXPRESSIONS

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Abstract: *Recent research studies have pointed out the crucial role of constructing and interpreting letters in algebra. Many difficulties emerge because of the incapability to relate the algebraic code to the semantics of the natural language.*
A teaching experiment, carried out with 16 year old students, attending the second year of the Gymnasium (a humanistically oriented High School) is described here. This experiment was aimed at analysing the cognitive behaviour of the students when facing learning situations dealing with a productive use of symbols and their understanding.
***Keywords**: algebra, natural language, symbols.*

1. The problem's Position

The general consensus considering symbols as a driving force of algebraic thinking has fostered a deeper investigation on the dialectical relationships between signs and algebraic ideas as well as a growing interest for teaching implications

There is evidence that the use of symbolic expressions can be a relevant cause of difficulties in students (see for example a study by Radford and Grenier, 1996). According to Laborde (1982) the development of a specific symbolic language can impoverish the meaning of the language previously used. If we look at the historical evolution of algebra, we see that rhetoric and syncopated algebra (i.e. algebra totally expressed by words and algebra expressed by a mixture of words and symbols) have been quite easy to use and understand. On the contrary, in a symbolic system the meaning of words and operations can stay behind the scene, since the symbolic language has the power of taking away most distinctions that are preserved by the natural language. Because of this specificity, the symbolic language expands its applicability but induces a sort of semantic weakness: it seems that such language is suitable for many contexts, without belonging to anyone. Hence the origin of the gap between symbols and meanings, which is confirmed by the rigid (stenographic) use of the algebraic code by many students. It often happens that some students are able to control the underlying meaning only when they make use of rhetoric or syncopated algebra.

The "rhetoric" method seems to be used spontaneously (Harper, 1987) and does not depend on the instruction level.

Research studies dealing with the process of constructing and interpreting letters (variables or parameters) point out its crucial role in algebra.

In this perspective the choice of names to designate objects is strictly related to the control of the variables which are involved to characterize the properties to be emphasized (Arzarello, Bazzini, Chiappini, 1994c). Difficulties emerge because of the impossibility to relate the algebraic code to the semantics of the natural language. The student able to express the relationships among the elements of a given problem correctly, by means of the natural language, can be unable to express the same relationships through the algebraic code.

The process of construction and interpretation is sometimes impoverished or blocked when the subject considers the terms in a rigid way, and does not grasp the underlying interrelation between sense and denotation of a given name (Arzarello, Bazzini, Chiappini,1994a,1995). In short, there is evidence that the student is often not able to take the whole potential of the algebraic code, i.e. the power of incorporating different properties within the name. The name is seen as a rigid designator, source of obstacles for algebraic thinking. Consequently, growing difficulties appear in front of algebraic transformations, and their additional requirement of foreseeing and applying, guessing and testing the effectiveness, in a continuous tension (Boero,1994).

All these issues foster a careful analysis of the questions related to the learning of algebra as a language; such questions are rooted, at an early school level, in the dialectic relation between semantics and syntax, procedures and structures, natural and symbolic language. The passage from natural language to symbolic language is a key point in the development of algebraic thinking and asks for special attention in teaching.

Furthermore, we have to take into account the outlined questions framed in the wider perspective of the use and abuse of mathematical symbols in school practice . Sometimes students' spontaneous symbols are not encouraged in school to develop towards generally accepted symbols. Hence, symbols are often the cause of many learning difficulties, because of their loss of meaning (see for example a study by Furinghetti and Paola, 1994).

2. The Teaching Experiment

A teaching experiment, carried out with 16 year old students, attending the second year of the Gymnasium (a humanistically oriented High School) is described here. The experiment was carried out in the period March-June according to the usual school year schedule: no additional lessons have been given to students. The experiment included a training phase (approximately three months), a break (two weeks) and a final test consisting of a questionnaire and clinical interviews.

The class was made up of 20 students at upper intermediate level. The mathematics teacher and an interviewer have been present in the classroom during the whole experiment, which was centred on the topic “Straight lines and linear systems”.

This experiment aimed at analysing the cognitive behaviour of the students when facing learning situations dealing with a productive use of symbols and their understanding.

The main objective is the study of the difficulties which persist in mastering symbols (namely the construction and interpretation of symbolic expressions) for students who have received a properly oriented training.

Since the previous school year, the teacher of mathematics had systematically been practising a meaningful oriented teaching, with special attention to the role of symbols. In each lesson a careful analysis of the symbolic expressions encountered have been done: the students have been requested to distinguish between unknowns, variables and parameters. They have also been required to reflect on the changes induced by the variation of a given letter.

At the end of the experiment, a questionnaire was administered to the students and, finally, each student was interviewed.

For our purposes, we will focus on two items of the questionnaire and two questions of the interview. Let's notice that the questions proposed (in the questionnaire as well as in the interview) are not routine questions, because they should test a genuine mastery of the symbolic code.

Here following the text of the two problems in the questionnaire:

PROBLEM I
Given the straight lines y=3x+q and y=mx+5
a) give a condition for being parallel
b) give a condition for being perpendicular
c) give a condition for having the point (1,3) in common
d) give a condition for having the point (0,0) in common.

PROBLEM II
A linear system of two equations and two unknowns, admits the couple (1,1) as a solution.
Discuss the following statements:
To get a solution which is double of the given one, (i.e. to get the solution (2,2)), one should:
a) multiply both sides of the two equations
b) multiply by two both sides of one equation
c) construct a different system.

2.1 Problem I: Analysis of the Students' Answers

Points a) and b) were correctly solved by 16 students (17 being the total number of students): just one did not give any solution.
The correct solutions at point c) were 10 (3 omitted and 4 wrong).
At point d) the correct solutions were13 (one omitted and two wrong).
There is evidence that the great majority performed well in this task.

As far as point c) is concerned, two students assigned arbitrary values to m and q, one worded a correct procedure but did not provide the numerical solution, another carried out a sequence of algebraic transformations unsuccessfully.

At point d) a student replied:"*No straight line passes through the origin*": it seems that the presence of the letter q "automatically" implies that the known term is different from zero. This misconception is not unusual: it appeared in similar conditions along the year. Other two students hold the belief that the straight line could pass, in some way, through the origin. More precisely, one student said:"*If m=5, then the equation y=mx+5 becomes y=x, i.e. a straight line through the origin*" Another student claimed:"*The equation y=3x+q passes through (0,0) when q=0; when the other equation too (i.e. y=mx+5) will have q=0, the two lines will meet in the origin*".

These kinds of responses clearly point out the presence of rigid designators (q different from zero and the possibility of manipulating formulas to be adapted as desired).

2.2 Problem II: Analysis of the Students' Answers

This problem was correctly solved by all the students. This confirms the good mastery of the algebraic meaning underlying a linear system and the operations which do not change the system's solution. Nine students out of 17 needed numerical trials. Five students used the geometrical referent: two replied that one needs to apply the translation (1,1) to both lines; three justified the incorrectness of the first two answers referring to the linear combination of the two equations: by multiplying the two equations by different factors, one gets lines still passing through the centre of the bundle.

Let's consider now two of the questions administered during the interviews:

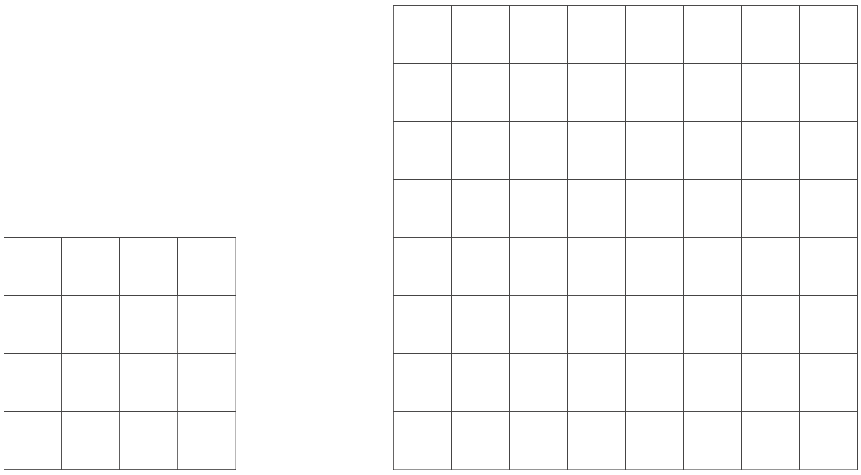
I) The cube problem
Try to express in an algebraic language that the ratio between the volume of a cube, whose edge is unknown, and the area of the square, whose edge is the double of the cube's edge, is two times the area of one face of the cube, which is augmented by three.

II) Which sense the following expressions have for you:
x^2=px+q y=mv+h

2.3 Question I): Analysis of the Students' Answers

Just five students out of 19 solved the question easily. More precisely, just one performed well, without any help: the others had some slips (i.e. $2l^2$ instead of $4l^2$ or $l^3/4l^2=2l^2+3$ instead of $l^3\,/4l^2=2(l^2+3)$. The remaining 14 students needed to read the text several times: probably they would not have been able to approach the problem without any help from the interviewer. Difficulties have been of a different nature: some due to deficiency in computation (for instance "ss=2s") or in the symbolic translation of simple sentences expressed in the natural language. For example a student asked if "*to augment by 3*" would mean "*to add 3*" or "*to multiply by 3*".

A student wrote "$V_c/A_q=3(2A_r)$"; another one approached the question by recurring to a numerical support, i.e. by attributing an arbitrary value (2) to the length of the cube's edge (and being consequently 4 the length of the edge of the square). Here the report of such trial: "*I have a cube of four little squares, the cube has six faces, hence 4x6, then I have another one with doubled edge ...*"



"*I have to compute the ratio of these two, i.e. 4x6.........*"
The interviewer underlines that they are speaking of a volume, and then she writes 4 and 2.

The interviewer asks "Why 2 and 4?" and she replies that the area of the square is l^2, the area of the cube's face is ½ l^2. She continues without any visible orientation:

½ l/l=2l+3 (½ l)* (½ l)=2l + 3 ¼ l = 2l + 3. and does not reach any plausible conclusion.

In this episode we observe that numbers are intended to support the formula to be constructed, but the trial fails.

Many students made the mistake of not using the parenthesis in the second side of the equation: 14 students wrote "... $=2l^2+3$" *instead of* "...$=2(l^2+3)$". Nine students expressed "*the area of the square having the edge doubled with respect to the edge of the cube*" as "$2l^2$" instead of $4l^2$ and they changed their writing only after the intervention of the interviewer.

Eight students did not express "*the area of a square having the edge doubled with respect to the edge of the cube*" or "*the area of the face of the cube*" as function of the cube's edge, but used several unlinked symbols. Here are some examples:

$$S^3 / 2S^2 = 2a^2 + 3$$

$$spig^3 : l^2 = 2\ (spig^2 + 3)$$

$$V/(2l)^2 = 2(l^2)+3$$

$$V_c / A_q = 2A_f + 3$$

$$\frac{l^3}{A_{\square} = 2l_{\square}\ 2l_{\square}}$$

These students created a new symbol (sometimes even figurative, i.e. $A_{\square}$; $2l_{\square}$) for each new element mentioned in the text, showing thus a very poor mastery of the synthetic function of the algebraic language. Finally a student wrote a totally out of focus expression: $\pi\, h / 4\pi = 2\pi + 3$

During the interviews, great difficulties emerged as far as the reading of the text was concerned: some students were able to memorize only a part of the text. Such difficulty is rooted in the incapacity to consider all things together: when a student forgets to consider a part of the problem, there is evidence of the difficulty in naming, which makes incorporating and clarifying the sense of a problem into the system of algebraic representation possible (see Arzarello, Bazzini, Chiappini, 1994b).

2.4 Question II): Analysis of the Students' Answers

The expression $x^2=px+q$ was considered as the equation of a straight line by 8 students out of 19. Other eight students interpreted the given expression as a second grade equation. One student replied. "*It does not remind me of anything*": after the intervention of the interviewer, he recognized that it could be an equation in p or q and, hence, a straight line. Just one girl gave a complete answer: "*It could be a literal equation of second grade in x, or an equation in the unknowns p and q, where x and* x^2 *are the known terms, or an equation in q (the only unknown),* x^2 *and px being the known terms*". Another student interpreted the expression in terms of objects: "*A squared thing, which equals itself multiplied by another thing, plus another additional thing*".

Eight students interpreted the expression as the equation of a straight line: one of them claimed "*It is a squared straight line*". Four students specified that they were not able to justify the presence of x^2, but this was not enough to contrast their wrong belief.

Eight students interpreted $x^2=px+q$ as a second grade equation: two of them noticed that if x^2 would have been substituted by y, they would have recognized a straight line.

This confirms the relevance, in the students' minds, of the pattern of straight lines.

Finally, an unusual behaviour of two students: they considered $x^2=px+q$ as the equation of a straight line in the unknowns p and q.

The expression $y=mv+h$ was considered by 9 students as the equation of a straight line, by eight students as a first grade equation.

Two students made reference to a formula constructed in a different context (not related to algebra or geometry), for example physics or chemistry.

It is worth noticing that five of the nine students who interpreted the expression as the equation of a straight line, clearly state that they were considering the expression as if v were substituted by x and h by q. Such students were flexible in seeing a straight line in the given equation, but they needed to refer to the standard writing $y=mx+q$ (rigid designator).

As observed, eight students saw a first grade equation: most of them considered the y as unknown. One student said it was a first grade equation in x (which did not appear): again the sign x as a rigid designator of the unknown.

3. Final Remarks

The experiment points out some key issues in the process of constructing and interpretating algebraic expressions.

It is clear that many problems are rooted in the students‘ incapability to construct suitable formulas, which should incorporate the meanings of the objects involved and their mutual relationships. This process is typical of algebra and constitutes a breaking away from arithmetic: it is something new to students, and even in the historical development of the discipline.

The experiment has shown that even bright students, who have received a good teaching, (that is oriented towards a meaningful understanding of symbols) and have performed well in the above mentioned problems of the questionnaire, encounter relevant obstacles when passing from rhetoric (or syncopated) algebra to symbolic algebra. In the change many students do not grasp the symbolic function of the algebraic language. As Norman (1987) pointed out, since the algebraic language is not directly associated to the semantics of the natural language, this latter can influence and distort the construction of formulas.

In fact, the choice of names to designate objects is linked to the control of the introduced variables: difficulties emerge because it is very unusual that algebraic formulas are a simple linear stenography of what is expressed by means of the natural language.

Consequently, the change from one semiotic register to another (natural language and symbolic language) does not occur appropriately and knowledge is not constructed (Duval, 1995).

As noticed by many authors, much work towards solution is done when a good process of naming has been produced by the subject (see, for example Chalouh&Herscovics, 1988; Kieran, 1989; Arzarello, Bazzini, Chiappini, 1994b).

The experiment has given further evidence of the students' difficulties in naming the elements of the problem in an appropriate way and coaching their stream of thinking in order to "condense" the most relevant aspects of the problem into a formula.

These persistent difficulties confirm our belief that learning algebra should be the result of a long cognitive apprenticeship during which the student learns to incorporate senses in algebraic terms and expressions, overcoming the stereotypes of rigid designators and eventually controlling the senses of the constructed expressions.

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FROM SENSE TO SYMBOL SENSE

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Abstract: *In this theoretical paper figurative aspects of algebraic symbolism are discussed and related to the theory of image schemata, opening up for one way of understanding the development of symbol sense in mathematics. The ideas of mathematical forms (referring to spatial characteristics of mathematical formulas), and form operations, are at the core of this analysis.*
Keywords*: -*

1. Introduction

The term ”symbol sense” has only recently come into use in mathematics education, referring to a similar kind of familiarity with algebraic symbols as ”number sense” to arithmetic: ”As students' understanding of algebra deepens, they are gaining *symbol sense*: an appreciation for the power of symbolic thinking, an understanding of when and why to apply it, and a feel for mathematical structure. Symbol sense is a level of mathematical literacy beyond number sense, which it subsumes.” (Picciotto & Wah 1993, p. 42) With a developed symbol sense as the main goal of algebra teaching (Arcavi 1994), a theoretical framework of the conception and its development is needed. The issue is complicated by the fact that algebra can be viewed both as a symbol system and as a way of thinking (cf Sierpinska 1995, p. 157). The interest here is on aspects of the symbolic language used in school algebra.

Leibniz, one of the most influential innovators of notations in the history of mathematics, wrote (see Cajori 1929, p. 184): “In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labour is wonderfully diminished.” This quotation (also quoted in Bergsten 1990), referring to notations in mathematics, indicates that when symbols in mathematical formulas are arranged in a way similar to those structures of the real world that the formulas are used to depict, then the notation is easy to understand and to work with. The idea that mathematical thinking, as well as understanding, as related to formulas, can be facilitated by observing the figurative characteristics of formulas, can find further support in the image schemata theory of Mark Johnson and George Lakoff, as noted by Dörfler (1991). In this theoretical paper some conceptions of mathematical form, introduced in Bergsten (1990), and in line with these ideas, are further developed and related to symbol sense. It should be pointed out that the space available for this conference paper does not allow a deeper elaboration of the ideas presented.

2. Mathematical Forms

When looking at strings or expressions of (standard) mathematical symbols (in elementary school mathematics) as pictures, only a very few basic patterns (spatial arrangements) appear. Bergsten (1990) thus defined *forms* as typographical units (*atomic forms* or *symbols*) or as spatial relations between typographical units (*molecular forms* or *patterns*). The notion of *mathematical forms* refers to symbols used in mathematics and patterns of such symbols. When combining atomic mathematical forms like 1, 2 and 3 to produce mathematical expressions like 213, 1+2, or 1+2=3, the molecular forms appear as the spatial characteristics (schematic structures) of the symbolic expressions. The forms here can be depicted by the schematic structures of the figures *III*, *IOI*, and *IOIOI* respectively. Seeing 1+2 as a unit (chunking) in the last of the expressions, the essential form is seen to be *IOI* there as well. This is the form of a *link* (that can be extended to a *chain* like in *IOIOI*), used in notations of both operational (1+2) and relational (1<2) ideas. The other molecular form that appeared here (i e *III*) has the pattern of *bars*. An algebraic expression like $3x + y = 5$ is a molecular form structured by the following “Chinese boxes”: (((3·x) + (*y*)) = (5)). It can be observed that the pattern is composed by a number of links only, in this case three.

Also when operating on mathematical symbols some basic schematic structures appear. Some of these are *splitting* (as in $3x \rightarrow x+2x$), *joining* (x+2x → 3x), *mirroring* or exchange of position (b+a → a+b, *adding* (3 → 30 when multiplying by ten), *deleting* (30 → 3 when dividing by ten), *raising* ($2x \rightarrow x^2$ when integrating), *lowering*

($x^2 \rightarrow 2x$ when differentiating). The schematic structures describing these symbol transformations can thus be seen as form operations and are composed into *form operators* (transforming for example an equation like $7x$-3=$4x$+6 into its solution x=3). Also the *identity* operation can be identified as a form operation. The role of form operators in algebra is essential. Pimm (1995) even argues that "the algebra takes place between the successive written expressions and is not the statements themselves." (p. 89)

Some basic forms seem to have their origin in bodily experience. These forms can be called *genetic* (Bergsten 1990), by which is meant that they can be isomorphically mapped onto pictures (or objects) of possible referents of the symbolic expressions. A simple example is provided by the expression 2+3, the form of which (link) also appears in the spatial arrangement of the picture oo ooo , showing the act or the idea of adding together two and three objects, a possible referent of the symbolic expression. The form of the algebraic counterpart a+b is inherited from the world of arithmetic, thus bringing along its genetic character. Forms that are not genetic are called *stipulated*, as most atomic forms (like 2 or √) or the form *superposition* used in exponential notation (as in x^2 or e^x).

Observations like these are of course trivial but seem to have far reaching implications for the understanding of (some parts of) mathematical symbolism:

- The three molecular mathematical forms just mentioned are not only commonplace, but practically the only ones that are used in (standard) elementary school mathematics to formalize the ideas of numbers and the four basic operations of arithmetic (forms for executing some algorithms excluded). They are also (notably as a consequence) commonplace in algebraic notation (including abstract algebra and vector algebra).
- The observation that genetic forms seem to be based on bodily experience invites the use of image schemata theory in discussing understanding and meaning of (some parts of) mathematical symbolism. In particular, the individual development of (some aspects of) symbol sense can be traced back to the structure of abstract thinking ultimitaley based on sense impressions.

- A structure-operation distinction within mathematical symbolism, based on the above definitions, provides a means of analysing the understanding of mathematical forms, using the complementary concept.
- The ability to link form and content in mathematics is commonly considered a main feature of what it means to understand mathematics (see e g Hiebert 1986). The dynamic interplay between these four dimensions of mathematics and doing mathematics (i e form and content, and structure and operation) gives a characteristic of the logical and psychological aspects of mathematics (Bergsten 1990).
- Bearing in mind that all discussions of learning and understanding mathematics in educational contexts by necessity must rest on some view of what mathematics is, the conception of mathematics as a *formalization* of ideas originated in our bodily experience or in other disciplines (like physics or economics), as elaborated by Mac Lane (1986), puts an emphasis on mathematical forms, as well as on its relation to the corresponding ideas and activities, when it comes to understanding.

The notion of genetic forms is related to the idea of analogical reasoning (Gentner 1989, English 1997), in the algebra context discussed in English and Sharry (1996). It is commonly argued that these ”surface” structures of mathematical symbolism are mere conventions, and that the meaning they carry must be found by the individual’s own mental constructions of their relationship to the ”deep” structure of mathematics, or the algebraic manipulations become meaningless (English & Sharry 1996). Here it is argued, however, that meaning often can be more or less directly evoked already at the ”surface” level by its very structure.

It should be observed that the notion of mathematical forms does not extend to a possible more abstract notion of ”mathematical form”, in the sense of general structures that are being formalized as mathematics. Such structures can to some extent be expressed by non-algebraic forms of representation, such as verbal and pictorial forms. The discussion in this paper is focussed on the implication for the understanding of school algebra of the notions of mathematical forms and image schemata.

3. Mathematics and Image Schemata

The conceptual framework of image schemata by Johnson (1987) and Lakoff (1987) has a tremendous potential impact on mathematics education. Its relevance to meaning in mathematics and mathematical reasoning and understanding has been pointed out by the authors themselves, and later, among others, by Presmeg (1991) and Dörfler (1991).

According to Johnson and Lakoff, meaning and understanding in human reasoning are based on the use of *image schemata*. Our bodily experience of ourselves and our physical environment, and our way of structuring and organizing this experience, establish cognitive image schemata (with a non-propositional character) that depict recurring regularities and structures of these experiences and organizing activities. These schemata thus are constructed by the individual. Examples of some basic image schemata are CONTAINER, BALANCE, PATH, LINK, SCALE, AND CENTER-PERIPHERY (see e g Johnson 1987, p. 21, 85, 113, 117, 121, and 124, respectively). A key notion in the theory is that these kinds of bodily based schematic structures also are used in human abstract thinking by the means of metaphorical projection from the world of bodily experience into the abstract dimension.

In her study on visualization in mathematical thinking Presmeg (1985) found two ways students used imagery to depict abstract situations, by *concretizing the referent* ("making a concrete visual image the bearer of abstract information"), and by *pattern imagery* ("embodies the essence of structure without detail"). The latter is a construct similar to image schemata (Presmeg 1991, p. 7).

Dörfler (1991), stressing the holistic aspect of (subjective) meaning of mathematical concepts, argues that the (mental) construction of appropriate image schemata (referring to Johnson and Lakoff) is one means of facilitating the "cognitive manipulation" of such concepts. This construction can be supported by working with protocols of actions.

According to Mac Lane (1986) activities like counting, ordering, shaping, and moving, give ideas of the common structures and regularities in these activities (across situations), which can be formalized as mathematics. Thus the activity of counting gives the idea of "the next one", which can be formalized as "successor" by using the

Peano axioms (giving a formalization of the mathematical "object" ordinal number). Behind mathematical concepts like limit and continuity is the idea of approximation, originated in activities like estimating. More complex mathematical structures seem to be generated from within mathematics itself. Thus it is appropriate to talk about *content-dominated* mathematics and *form-dominated* mathematics (Bergsten 1990), thereby indicating that the mathematical forms used in the former have a more direct link to ideas based on bodily experience, or, in the above terminology, are predominantly genetic.

Many mathematical notions can be related to ideas of an image-schematic character. Set theory can be seen as the formalization of properties of the CONTAINER schema, distance as the CENTER-PERIPHERY and SCALE schemata, a function as the LINK schema and its graph as the PATH schema, and an equation as the BALANCE schema. Many concepts and methods of mathematics seem to be developed from the human way of thinking by the use of bodily based image schemata. Whatever the "real" mechanisms behind the historical development of mathematical notions and notations have been, the use of image schemata theory when discussing mathematics seems to have a high potential value in the educational context. Dörfler writes (1991, p. 20): "The cognitive manipulation of mathematical concepts is highly facilitated by the mental construction and availability of adequate image schemata. In other words, the subjective meaning of mathematical terms has a non-verbal, non-propositional and geometric-objective component. The individual understanding of a mathematical topic possibly is best grasped as a kind of interplay between the propositional expressions and corresponding image schemata." The last sentence from this quotation can possibly be seen as an image-schematic interpretation of the above mentioned (common) view that much of mathematical understanding relies on making connections between form and content. The (subjective) meaning of mathematical content then is given by its image schematic structure, a view in line with Mac Lane's description of mathematics, as indicated by Lakoff (1987, p. 365, referring to an article by Mac Lane from 1981): "Mac Lane's view of mathematics is thus very much like the view of human conceptual systems that has emerged in this book." The forms of mathematics "are those that emerge from our bodily functioning in the world and which are used cognitively to comprehend experience." (Lakoff 1987, p. 365)

These ideas have recently been further developed by Lakoff and Núñes (1997), who set up the new task of describing the ”metaphorical structure of mathematics” (p. 21), and they write: ”One of the properties of commonplace conceptual metaphors is that they preserve forms of inference by preserving image-schema structure.” (p. 30)

4. Mathematical Symbolism and Image Schemata

The relevance of the image schemata theory to (the understanding of) mathematical symbolism has been noted by Dörfler (1991, p. 28): “I think that formulas in mathematics can play the role of a carrier for an appropriate image schema. The latter then is made up by the spatial relations of the symbols in the formula and of the admissible operations and transformations with the formula. It is this image schema which lends meaning to the formula as its concrete carrier”. Here similar structural (figurative) and operational aspects of mathematical symbolism as those introduced in Bergsten (1990), and discussed above (in the paragraph on mathematical forms), are stressed.

With this focus on the figurative aspects of mathematical symbolism, a key observation is that genetic mathematical forms (as defined above) show the same schematic structures as the BALANCE schema, the LINK schema and others, and this almost by definition! That forms (schematic structures) are genetic means that they are based on the way humans structure and organize their bodily experience. A possible redefinition of the notion *genetic* mathematical form, keeping the intention behind the original definition intact, relates such a form to a (similarly structured) prototypical image schema behind the idea of the referent of the symbolic expression. Now, image schemata are subjective, but this redefinition makes sense, since many basic schemata are prototypical and socially shared.

When an idea is based on the activity of putting things together, it (by human imagination) can be mentally structured by the LINK schema, by Johnson (1987, p. 113) schematically materialized by a picture like o-o . That the recording in some notational form of such an activity or action (what Dörfler calls protocol of an action), shows the same schematic structure as the image schema itself, is more likely to be expected than not. The historical success of the mathematical form used in the standard notation for addition, *a+b*, thus has a psychological explanation in its prototypical image schematic character.

Johnson (1987, p. 90 and pp. 95-98) suggests that the BALANCE schema, with its properties of symmetry, transitivity, and reflection, provides a possible experiential basis for the ideas behind the mathematical concepts equality and equivalence relation. In the schematic structure of the symbolism, the equality sign (in itself made up by two equal parts, as stressed by Robert Recorde himself) serves as the fulcrum of the balance. In the terminology above, the mathematical form of written equalities is a link (when chunking of left and right “wings” is made), and must be classified as genetic by its connection to the BALANCE schema.

Form-dominated mathematics naturally inherits some of its mathematical forms from the content-dominated mathematics, making it possible to “understand” such forms by metaphorical projection from the image schematic structures of genetic forms. The notation for associativity of compositions of elements of groups, a purely formal property, is (most likely) inherited from the notation for repeated addition (or multiplication), exhibiting a genetic mathematical form. The ease of operating on such purely formal expressions, as well as the sense of understanding (in some way) what one is doing, can be explained by the image schemata that are being evoked by observing the schematic structures of the formal expressions.

Dörfler’s (1991) recommendation to use protocols of actions (to promote the construction of adequate image schemata) to develop understanding of mathematical concepts genetically related to those actions, thus also applies to develop understanding of mathematical forms. The common element is the construction of image schemata. In the case of forms, however, the actions of interest are the reading, manipulation, and interpretation, of (mathematical) symbolic expressions.

The notion of mathematical forms brings the focus to the structural (figurative) aspects of (mathematical) symbolic expressions. Form operations deal with the operative aspect of symbolic expressions, transforming them to new symbolic expressions. Some common such operations have been identified above, and their image schematic character is based on bodily experience. No doubt, activities like splitting or joining (putting together), adding or deleting (taking away), and raising or lowering, are prototypical across many experiential situations from low ages on. Also the identity operation can be experienced, as when turning a cube upside down or rotating a ball, leaving the object "unchanged". It makes sense to refer to some form operations as *genetic*, as splitting, joining, and mirroring, on some symbolic expressions in arithmetic (see examples above), while others, like adding, deleting, raising, and lowering (as exemplified above), are *stipulated*. Using Dörfler's (1991) terminology, image schemata related to mathematical forms are *figurative*, image schemata related to form operations are *operative*.

5. Developmental Aspects of Understanding Forms

Real numbers are used ("applied") in many different kinds of settings: though dealing with a variety of situations (denoting magnitudes, proportions or time) they still obey the same formal rules. To "understand" what real numbers are then by necessity involves "understanding" the formal rules for such numbers. These formal rules are "visible" in mathematical formulas such as $a+b=b+a$. This means that for a learner, understanding of mathematical forms (in the sense defined above) is a path that (possibly) can lead to an improved understanding of mathematics. Indeed, genetic forms like in the above formula (for commutativity) have the potential of evoking the same image schemata as the ideas or activities they depict. This way a link between form and content can be (mentally) established, and the idea of understanding formulas as "pictures" of image schemata can be evoked. To operate on formulas of genetic form is like operating directly on the reality. They are isomorphic activities. This observation may be one key to answering the classical question on why mathematics works, i e that results of symbolic manipulations are applicable to reality (Bergsten 1990, p. 166; cf e g Kline 1985).

Thus, in mathematical understanding there is a dynamic interplay between form and content, facilitated by the use of image schemata. On the other hand, there is an interplay between structural and operational aspects of form as well as of content. In her analysis of conceptual development, Sfard (1991) stresses the complementarity of the structural and operational aspects of mathematical concepts and entities. Understanding the structural aspect of a concept presupposes the use of operational aspects of concepts with a structure already understood. However, there is a vicious circle: "*the lower-level reification and the higher-level interiorization are prerequisite for each other!*" (Sfard 1991, p. 31). This can explain, Sfard argues, why for many students the whole enterprise of mathematics can become a meaningless game of rules producing correct answers. The reason is that reification has not occurred. Seeing a structure is more difficult than performing an operation. Therefore many students need well-planned educational activities for the reification process. The use of protocols of actions to produce adequate image schemata (Dörfler 1991), has the potential of facilitating this process.

Understanding mathematical forms seems to develop in a similar way. The schematic structure of a formula can become visible by operating on instances where the formula holds. For the "reification" of the form (of the formal notation) for commutativity, for example, adding different pairs of numbers in both orders and systematically recording these operations symbolically (e g in columns), the schematic structure of these notations can be interiorized, performed and "seen" purely mentally, and finally reified as a formal structure. In this case the known form used is the *link* form of the addition formula, and the new form produced the form operation *mirroring* for addition formulas. The sense of understanding these genetic forms is hypothesized to be given by the similarly structured image schemata they evoke, originated in bodily based activities and experiences.

Now, there is no need to understand the *formula* for commutativity (not to be mixed with the *idea* that adding in reverse order yields the same result) until it must be used in a formal computation. In school this normally happens only after algebraic notation has been introduced, and that is also when understanding mathematical forms is beginning to become crucial also for mathematical achievement. For some students the algebraic terms themselves are void of meaning: they are simply symbols (atomic forms) like a, b, and c, in written (rule-governed) expressions like $a+b=c$. Thus in beginning algebra there is a risque for a *double meaninglessness*: operations are performed on algebraic expressions, made up by terms without meaning, and with a form without meaning.

To develop an understanding of mathematical forms, and pave the way for symbol sense, its structural and operational aspects must both be grasped. These seem to presuppose each other, i e are complementary for this understanding. To apply the form operation joining on the left wing of the expression 7*x*-*x*=54, to obtain 6*x*=54, the schematic structure of the expression must be understood (possibly by the LINK and/or BALANCE image schemata). On the other hand, to see the basic structure, the link around the equality sign, it must be observed that a joining operation can be performed on 7*x*-*x*.

6. Symbol Sense

Some aspects of symbol sense in the context of school algebra have been described by Arcavi (1994), one of which is strongly related to the present discussion: ”[What is symbol sense?] - An ability to manipulate and to ”read” symbolic expressions as two complementary aspects of solving algebraic problems.” (p. 31) It is argued in this paper that the development of this (key) aspect of symbol sense can be facilitated by building on the recognition of the role of image schemata and genetic mathematical forms for the feeling of understanding the mappings between form and content in school algebra. These constructs provide a link from sense impressions to the development of symbol sense in elementary school algebra.

Higher-level symbol use in (form-dominated) mathematics rests heavily on familiarity with the basic algebraic language used in school. The role of non-propositional, imagistic thinking is furthermore prevalent at all levels of mathematical thinking (cf Sfard 1994).

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THE ROLE OF ALGEBRAIZATION IN THE STUDY OF A MATHEMATICAL ORGANIZATION

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Abstract: *Algebra is considered here, not as a particular mathematical organization (such as arithmetic or geometry, for instance) but as a process, the process of algebraization, that can affect either a whole mathematical organization or, as is nowadays the case in secondary education, some aspects of it. Depending on the case examined, a more or less algebraized mathematical work results: the algebraization process can in fact be considered as the modelling of a whole initial mathematical work in order to deepen our knowledge of it. Algebraization is thus, in this sense, a specifically didactic activity, relative to the study of mathematics. Institutional restrictions on the algebraization process constitute a paradigmatic example of the fact that mathematics (the object of study) and didactics (the process of study) are inseparable.*
Keywords: *algebraization, process of study, mathematic organisation.*

1. Mathematics as the Study of *Mathematical* Works

In their daily work as teachers, students and researchers, mathematicians usually describe mathematical knowledge in terms of concepts, notions, intuitions, ideas, methods, processes, definitions, problems, etc. They can then talk about ”students' difficulties in the acquisition of a concept,” or about the ”inability to apply a notion correctly in order to develop a new method,” ”the importance of being able to formulate a good definition,” ”the danger of focussing on processes while disregarding the general idea hidden behind them,” etc. What is used, in all these cases, is the *common epistemological model of mathematics*, that is, the usual way of perceiving, interpreting, describing, and thinking about what mathematics is and what mathematical activity exactly consists of.

The anthropological approach to the didactics of mathematics (Chevallard 1992) establishes that *institutional mathematical practices* are didactics' primary object of study, which means that, in an initial research stage, mathematical activities cannot be taken for granted, nor considered as a primitive concept but, on the contrary, as entities that, just like any other object of scientific study, need to be questioned and modelled. In order to achieve this, the anthropological approach proposes a general epistemological model that describes *mathematical knowledge* in terms of *mathematical works* or praxeologies, and presupposes that these mathematical works are structured into institutional mathematical organizations, which, in turn, are made up of four main components: several types of (problematic) tasks, techniques (to carry out these tasks), technologies and theories. The pair [task/technique] constitutes the praxis (or ”know-how”), whereas the pair [technology/theory] is the logos (or ”knowledge”) of the whole praxeology, which includes both praxis and logos. This theoretical framework also describes the process of creation of these four components, that is, the way in which a technique can emerge from the study of a problematic task (or ”problem” for short), its development into new techniques, the way in which certain descriptions and justifications of this work become a technology (i.e., a discourse —logos— about a technique —technè—), etc. (Chevallard, Bosch & Gascón 1997; Chevallard 1997).

Generally speaking, mathematical activity can be considered as the use of a mathematical organization or a mathematical work. But it is also, at the same time, a production (or re-production) of mathematical realities that will lead to new mathematical organizations. The English term ”work” (translated form the French œuvre) allows us to talk about mathematics as a human activity —given that mathematics are something we do— and as an artifact produced and reproduced by this activity —the work of mathematicians—. A mathematical work is something to be used and something to be produced or reproduced. Taking the word ”study” in its broadest sense, the anthropological approach considers mathematical activities as processes of study of works within given institutions. This process of study —or didactic process— includes mathematical research, which always begins with the study of open questions and problems, as well as the so-called teaching and learning process. The didactic process takes place in a community (groups of researchers, of students, etc.), is usually supervised by a director of study (be it the main researcher, the teacher, an advanced student, etc.), and has its own program of study (an open problem, a whole research program, the curriculum, etc.).

According to the four-component structure of mathematical works, the process of study can be described in terms of six didactic moments or dimensions: the moment of the first encounter, the exploratory moment, the moment of the technique, the technological-theoretical moment, the institutionalization moment, and the evaluation moment. It is important to note in this respect that we speak about ‘moments’ in a functional, not chronological, way, as when we say that something is bound to happen or that ”there is always a moment when ...”.

Within this theoretical framework, we would like to emphasize a rather surprising aspect of the link existing between didactics and mathematics. We start by considering the algebra related to the algebraization process of a mathematical organization. This process can be defined as a process of mathematization which consists in the construction of a new mathematical organization that models the given one. We therefore do not consider school algebra as a mathematical organization in itself, but as a way of modelling a given mathematical organization so as to make its study easier. We can talk, in this sense, of a didactic technique, that is, a special *way of studying* a mathematical organization. This point of view allows us to analyse the consequences that the algebraization process has on the study of a mathematical organization and to stress a number of institutional restrictions imposed on the process of study of mathematics in secondary education.

2. The Need of a Model of School Algebra

In Gascón (1994) we analysed the didactic phenomenon of school arithmetization of elementary algebra and related this phenomenon to the interpretation —which is rather common in pre-university academic institutions— of algebra as 'generalized arithmetic' (in the sense expounded in Booth 1984, Drouhard 1992, Filloy & Rojano 1989, Kaput 1996, Vergnaud 1988). We also maintained that it is necessary to go beyond the mere questioning of this interpretation to take it as an object of study, i.e. as an empirical fact that didactics should be able to explain. But this can only happen if, on the one hand, we establish a notion of algebra that will allow us to interpret 'the study of algebra' in a given institution, and, on the other hand, we use it as the basis to generate a series of didactic phenomena related to what is commonly called ”the learning process of algebra.” Contrary to any given naturalistic point of view, these phenomena, just like psychological, sociological, historical or physical phenomena, have to be constructed scientifically and cannot be taken for granted.

Elementary algebra does not appear as a self-contained mathematical work comparable to other works studied in academic core courses (such as arithmetic, geometry, statistics, etc.), but rather as a modelling tool to be (potentially) used in all mathematical curricular works and which appears to be more or less used in them. According to that, the model of elementary algebra that we have chosen as an alternative to 'generalized arithmetic' is based on the realization that elementary algebra is in fact a mathematical tool, the algebraic tool, that can be used to study many different kinds of problems not only or exclusively pertaining to arithmetic, which leads to the notion of algebraic modelling as described by Chevallard (1989). We will therefore not talk here about the teaching of 'algebra' as we do about 'arithmetic' or 'geometry'. We will consider school algebra, not as an organization that can be studied per se, but rather as a mathematics set-up that makes the existence of a given process —the process of algebraization— possible, and which does not affect the mathematical works studied at school always in the same way.

Since mathematical activities (including school mathematics) appear, from a given stage of development, as being fully algebraized, in the sense that they cannot be conceived of without the whole functionality ascribed to the algebraic tool, we have to admit the existence of a process of algebraization of school mathematics. This process starts with primary education, continues throughout secondary education and is completed at the university level. Although this process may take place, on certain occasions, in an explicit way that can be recognized by the actors of the institution, it emerges more often than not in a subtle, surreptitious way, depending on the mathematical needs arising at any given moment, the actors of the institution usually becoming aware of its existence without even having witnessed its actual emergence. In secondary education, mathematics tends to have a clear 'pre-algebraic' nature, as if it remained foreign to the algebraization process and would not even allow this process —which happens to be unavoidable— to take place in a coherent and controlled way, in accordance with its corresponding level (Gascon 1998).

Our main purpose is to start describing, analysing and characterising the multiple links existing between the degree of algebraization of a mathematical organization and the possible ways of conducting its study. We shall also show, reciprocally, that the different kinds of constraints of a mathematical, didactical and cultural nature which are imposed on the process of algebraization of mathematical works at school constitute, in turn, restrictions to the didactic process, that is, to the specific way in which these works can be studied at school.

3. The Algebraization of a Mathematical Work

It is important to point out that we do not have any demarcation criteria to delimit precisely what an algebraized work is and differentiate it from a pre-algebraized one (in the sense of ”not yet algebraized”). We thus postulate that there is always a question of degree in the algebraization of a mathematical work, and we will try to characterize this process by means of the notion of *modelling* (Chevallard 1989). More specifically, we will state that a *mathematical work is algebraized* if it can be considered as an *algebraic model* of another mathematical work, the *system to be modelled*.

Such a definition clearly transfers the problem in question to the characterization of algebraic modellings. Although we cannot offer a complete description of them here, we can at least present some of their characteristic features, which will allow us to distinguish algebraic modellings from other kinds of mathematical modellings:

(3.1) *The algebraic modelling of a given mathematical work describes explicitly and materially all the techniques contained in the initial work*, thus allowing for a quick development of these techniques, as well as for the explicitation of their interrelations and the unification of the related types of problems.

(3.2) *An algebraic modelling can be considered, in this sense, as the answer to a technological questioning related to the initial work*, such as, for instance, the way in which to describe and justificate the initial techniques, the conditions under which they can be applied, the types of problems they can solve, etc.

(3.3) *In an algebraic modelling, all components of the initial work are modelled as a whole*, and not as separate entities, a fact which tends to simplify the structure of the algebraized work eventually obtained.

EXAMPLE: Let us consider, as pre-algebraic work to be modelled, *the usual mathematical organization built up around elementary divisibility problems*, such as: how to find the multiples and divisors of a given number, how to set up divisibility criteria, how to calculate common multiples and divisors of two or three numbers, the greatest common divisor or least common multiple of two or three numbers, the list of the prime numbers down to 100, etc. The associated techniques are based on the fundamental properties of multiplication and division of integers, which make up the core of the technological-theoretical component. These techniques take the form of numerical tables which contain the multiples of two numbers or the multiples of the smaller number and the difference between the first two, the divisors of two numbers, the successive differences between two numbers and between the smaller one and the difference obtained, etc. (see below) and are bound to be developed until the standard form of Euclid's algorithm is obtained.

COMMON MULTIPLES OF 240 AND 300:

240	60 (=300-240)	300
480	120	
720	180	
960	240* (rep.)	1200 (=960+240)
1200	300	
1440	360	
...	...	...

G.C.D. OF 551 AND 437:

551	437	551-437=114
114	323	323-114=209
114	209	209-114=95
114	95	114-95=19
19	95	95=19ÿ5
19	0	19 = (551,437)

The *technological questioning* of such a work affects the description, justification and scope of the techniques in use. One may begin, for instance, with the following questions: Why does a common multiple of two numbers turn out to be a multiple of their difference? Why is the first repeated multiple in the table the least common multiple? Why calculate differences when looking for a common divisor? How does this relate to Euclid's algorithm? How can we interpret partial results of this algorithm? In which cases can we simplify it? etc. Algebraic modelling of the arithmetical techniques based on numerical tables should lead to the consideration of *linear systems of Diophantine equations*. Thus, the above tabular technique for the determination of the l.c.m. of 240 and 300 can be modelled with the linear Diophantine equation $240x + (300 - 240)x = 300x$, which determines a common multiple of 240 and 300 if we can find an integer y such that $(300 - 240)x = 240y$, that is $60x = 240y$. The solution $x = 4$ and $y = 1$ gives the l.c.m., i.e. $240·(4 + 1) = 300·4$. This kind of modelling allows the description of the original types of problems and the explicitation of possible links between them (specially between the l.c.m. and the g.c.d. of two numbers). It can also answer the technological question mentioned above and contribute to the formulation of new types of problems (for instance, which conditions have to verify two numbers a and b if they have 24800 as l.c.m.?, which common divisor of 120 and 200 is a multiple of 6?, etc.). Finally, the model based on linear Diophantine equations both unifies and simplifies the structure of the algebraized organization, and it also produces a clear ostensive reduction of the scripts used.

4. Characteristics of an Algebraized Mathematical Organization

Having already described an algebraized mathematical work as being the result of some algebraic modelling, we shall now present a few indices of the degree of algebraization of a given mathematical work:

(4.1) In algebraized works, modelled techniques take place at a technological level with regard to the original pre-algebraized work. An index of algebraization of a given mathematical work is therefore linked to *the possibility of formulating and studying technological questions* related to the description, interpretation, justification and validation of the initial work.

(4.2) In particular, the more a mathematical work is algebraized, the more it enables us to *describe the different types of problems that can be solved, as well as the necessary conditions for solutions to exist, their possible uniqueness and their structure*. In an algebraized organization, the main type of problems is no longer the determination of solutions but the study of their conditions of existence. Consequently, problems will not be studied in themselves, as isolated entities, but as components of a given type of problems.

(4.3) A mathematical work is constructed to answer some type of problematic questions that, in turn, give rise to various types of problems. Thus, *an indicator of the algebraization degree of a given work is linked to the possibility of considering, describing and handling the global structure of the above-mentioned problems*.

(4.4) *In an algebraized work, we use parameters and variables systematically*, both in mathematical techniques and in the associated theory. This use includes the manipulation of formulas and is supposed to give rise to a functional language.

(4.5) An algebraized mathematical can easily become independent from the original system that it originally models. More specifically, *the possibility of generating problems that are detached from their context of emergence constitutes an indicator of the degree of algebraization of a given work*.

(4.6) The algebraization of a work is also related to *the unifying of the different types of problems that are contained in this work, as well as to the integration of its corresponding techniques and technological elements*.

(4.7) This simplification of the components of a given algebraized work leads to a considerable reduction of the 'ostensive' material (such as words, writings, graphics, gestures, etc.) that is used to develop mathematical activity. In other words, an algebraized mathematical activity needs less 'ostensive material' to be worked out, but a material which is much more instrumental (Bosch, 1994).

5. The Role of Algebraization in the Study of a Mathematical Work

It is essential, in this context, to try and analyse the actual or potential consequences of the algebraization of a given mathematical work as regards the many possible ways in which this process can be conducted in modern educational institutions. These consequences need to be confronted to the *institutional didactic contract* that currently governs the study of mathematics at school. We will be using here, to this effect, the six dimensions of the process of study involved in the anthropological approach to didactics.

(5.1) In the process of study of a given mathematical algebraized work, *the exploratory moment has a necessarily 'material' nature*. Indeed, exploration is not carried out thanks to a purely 'mental' process, but rather with the crucial participation of handwriting. In other words, most of the ”plausible reasoning” (in the sense of Pólya) is carried out through calculation. More than any other, the algebraization process reminds us that *mathematics is a 'material' activity*, that it cannot be achieved without resorting to material instruments (be them oral, written, graphic, or ‘gestural’). This characteristic aspect of algebraized mathematical activity has to be contrasted with the currently existing cultural illusion —reinforced by the dominance of pre-algebraic ”mathematical thinking”— that mathematical activity takes place ”in our mind” only.

(5.2) If we want students to be able to develop by themselves the techniques they use, instead of having these techniques presented to them by the teacher, then the process of study of a mathematical organization is to go beyond the exploratory moment and the routine application of ready-made techniques. A *'material' representation of these techniques will then be needed to permit their explicitation and to start the technological questioning about their scope, interpretation, and justification*. We can then say than the algebraization process not only emerges from a technological questioning about a previous given work, but that it also allows this kind of questioning. *In other words, the technological questioning about the description, interpretation and validation of a given work appears closely and doubly linked to the process of algebraization of this work*. Difficulties about bringing into being the technological-theoretical moment are coherent with the pre-algebraic nature of the mathematical works studied in secondary education, and, consequently, with the obstacles to the complete algebraization of these mathematical works.

(5.3) In the process of study of a given algebraized work, the above-mentioned technological questioning leads to the *establishment of fluid interrelations between the moment of the technique and the technological-theoretical moment*. The flexibility characterizing the techniques used in this process is thus reinforced, and so is the power inherent in the moment of the technique *to integrate the different components of the mathematical organization* in question. In the study of a pre-algebraic mathematical organization, such relations are more difficult to establish, which produces a certain rigidity in the use of techniques and impedes the study of the mathematical organization as a whole.

(5.4) Pre-algebraic mathematical activity always implies a certain 'atomization' of its components due to the fact that the cultural interpretation of 'concrete' systems, as well as the apparent 'naturalness' of pre-algebraic techniques (especially arithmetical and geometric ones), provide in advance the necessary intelligibility for its survival in the institution where this activity is bound to take place —even though, in most cases, this intelligibility might turn out to be purely 'local'—. On the other hand, it is to be noted that *any algebraized organization is bound to lose a considerable amount of cultural intelligibility*, while preventing the activity in question from being atomized, being as it is the result of a *sustained and continuous process of study* that is in constant need of interpreting and evaluating the original organization globally. Again, the pre-algebraic nature of mathematical organizations studied at school is coherent with the absence of medium-term and long-term didactic goals, as well as with the difficulties inherent in the didactic process as regards the institutionalization and evaluation of a whole mathematical organization.

We have emphasized the strong relationship between the structure of the mathematical organizations liable to live in a given didactical institution (such as secondary education) and the kind of didactic processes that can be made to exist in that institution. More specifically, we have shown that the algebraization of a mathematical work can be considered as a didactic tool that facilitates the process of study of a mathematical organization as a whole, and leads to what we will call the *integrated study* of a mathematical work. Starting from this point, many questions should be raised, especially those relating to the conditions under which the ”algebraic-didactic” technique can be used in the classroom. What are the most appropriate pre-algebraic mathematical organizations to be modelled? What kind of institutionalization should they undergo to ease their algebraization? How to induce a technological questioning about these organizations? At what level can we consider the explicit algebraic modelling of mathematical techniques as an object of study? In short, to what extend does the kind of didactic process conducted in secondary education reinforce or, on the contrary, prevent the full algebraization of mathematical works?

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SIMULATION OF DRAWING MACHINES ON CABRI-II AND ITS DUAL ALGEBRAIC SYMBOLISATION: DESCARTES' MACHINE & ALGEBRAIC INEQUALITY

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Abstract: *This paper focus on the connection between geometrical constructions and algebraic descriptions derived of simulate some mathematical machines in CABRI-II learning environment. We present a scenario experimented with pupils in a first degree of a high school (eleventh grade). The action's motive is modelling a Descartes' machine as a CABRI-II diagram. The starting hypothesis leads on the fact that such situations promote connections between geometrical properties and their dual algebraic symbolisation. The animation of CABRI-II diagrams provides the visualization of geometrical aspects, and calculus on geometrical objects provides the algebraic symbolisation. This inquire into the meanings emerged of this mathematical experience can be completed by asking the pupils how to refute (using the geometrical tools and objets experienced) a frequent statement about the usual algebraic inequality at this academic level.*
Keywords: *algebra, dynamic geometry, microworld teaching.*

1. Introduction

About the signification of algebraic variables

In a previous study (Hoyos 1996 and 1998), we have identified several significations that high school students (in Mexico from 16 to 18 year old) attach to linear equation after having followed a traditional education course of analytic geometry (according to a text like Fullers' 1979). We have documented[1] with the case study the importance that the pupils attach to the operational character of the algebraic equations, the one which expresses the form ax + by = c.

Besides, other authors (Schoenfeld et al., 1993; Duval, 1988; Herscovics 1980) had already emphasized on the difficulties the students have when they are confronted with establishing links between the graphical and the algebraic representations, especially for the equations lineaires with two unknown. In the work we are presenting here, we propose simulation's scenarios on CABRI-II which are seeking for the links' establishment between the usual graphic and algebraic representations in high-school (eleventh grade). In the referenced studies (Schoenfeld et al. with the implementation to the software GRAPHER, and Duval with the proposal emerged of didactical analysis) these authors have also looked for the establishment of links between the graphic and the algebraic representations. In this pursuit, we try to contribute to the consideration of the algebraic equation's emergence on an epistemological point of vue, like the dynamic plot of curves and the Thales theorem's application.

2. Class Room Context

This presentation shows a short part of a function's teaching process in a eleventh grade school class in France. The all process can't be described here but we can give nine main points studied in the class room.

1. Variable notion (dynamic variable visualisation with counter and axis in Cabri II)
2. Functional dependance and proportionality
3. Different frames (geometry, mechanics..)
4. Graphic construction and interpretation related to geometrical problems in dynamic geometry.
5. Graphic frame (graphic's geometry)
6. Equations and inequations solving with graphic method.
7. Use of Algebra in analytical geometry (Descartes' Machine, Circle equations).
8. Study of classical functions (French curriculum)
9. Evaluations.

In all the process students use classic methods with paper and pencil but also a part important of graphics with graphic calculators and with the Cabri-geomètre II software[2] . The process was used in classroom from September 1997 to June 1998[3]. All students knows the software Cabri-geomètre by using it in the mathematical course for geometrical studies since the beginning of year.

3. Theoretical Frame of our Educational Experience

Descartes' voice in the classroom

We present here the introduction of the simulation on CABRI-II of Descartes' machine[4].

This machine, to plot hyperbolas, is made of a mechanism, composed of a system (RSQ) of a fixed slope that slides vertically and of a shaft (AF) tied to this system. This shaft (AF) is also articulated to a fixed point (F) on a horizontal axis (see fig.1). By introducing the simulation on CABRI-II of the mechanical construction of the plots, we wish to get close to apply the fundamental elements of Descartes' analysis situations in order to obtain algebraic equations.

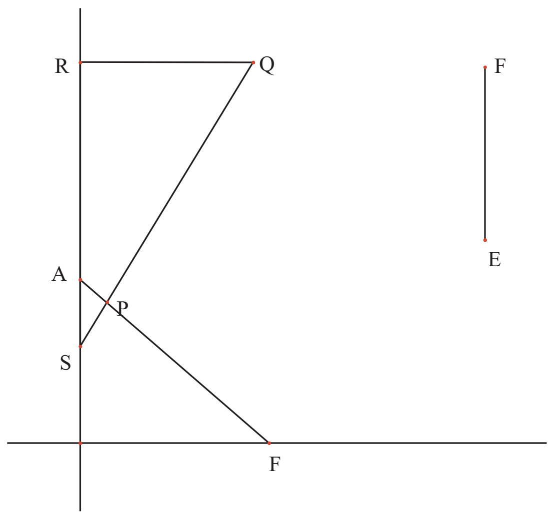

Fig. 1

Our teaching proposal has been complementary of the normal mathematics course of the first year[5] of high-school (eleventh grade class). We set two practical work (PW) up, one of which (the first) was constituted by three sessions of practical work of one hour each.

The task's sequence of this first PW has been:

1) Modelling the machine.
2) Algebraic characterization of the straight line's families that step in the plot of Descartes' hyperbolas
3) Obtaining the equations of hyperbolas.

We complete this instructional work with other activities (statement of the theme and graphic representation of the functions) organised around the refutation of one of the solution's procedure (specially the crossed product and the table sign's change) of algebraic inequality, usual for the pupils and yet incorrect. To resume, we have introduced a « Descartes' voice » in the classroom of eleventh graders, looking for a production of « echos[6] » that could lead to the establishment of geometrico-algebraical links. All the produced voices would give resources to the pupils to validate algebraic executions usual at this academic level.

4. The Sources and the Techniques

Descartes' machine

In the book II of *La Geometrie* , Descartes presents a method of curves' generation according to which we can obtain a curve's family to a given algebraic curve.

His procedure was as follows: we start from a previously constructed curve SQ, S a fixed point on the curve and A a point that is not on the curve; both points are fixed regarding to the given curve. Be O, a fixed point on the straight line RS, and F, a fixed point on the line perpendicular at the straight line RS in point O. Be P, an intersection point of the curve with the straight line FA. So, when the curve QS (also A) moves with a rigid translation movement parallel with RO, the point P will plot a new curve PF, which can be considered as the daughter of the original curve. Thus, if the given curve SQ is a straight line, the new curve will be an hyperbola. Here (fig 1), we can see a simulation of this machine on CABRI-II.

Mathematical experience in CABRI-II

We started by showing a video (5 mn) which we especially elaborated (and which is available from the EIAH[7] team) to introduce the theme of drawing machines. In this video, we talk especially about mathematical experience, specifically practice or the use of tools to do mathematics, and we show the advantages of the simulation on CABRI-II of this kind of machines. We asked then pupils to modele in CABRI-II Descartes' machine, after seeing it in the video.

Indeed, the students (working in CABRI-II) committed them selves in the proposed experience.

The sessions of first practical work (PW), and the results:

1) The first session of this PW aimed to the simulation on CABRI-II of Descartes' machine, the one the video just showed. For this construction, we had given the following instructions with figure 1 :

> *"Descartes' machine is constituted of two perpendicular lines and of a right-angled triangle that slides on the vertical line when we move the point S. During the moving, the triangle does not deform. The points R and Q cannot be directly moved. A is a point of the segment RS which position can be modify. The point F is a point of the horizontal line we can move. The length of the [DE] can be modified and allows to adjust the length of [RQ]. Once the machine is made, use the tool trace for the point P, and drag S to see the curve plotted by the machine. Save your machine in a file.*

The students were very interested in the task, and half of the pupils succeeded in the modelling during the planned time session (approximately 50 mn). Between the pupils' approaches to note, there is the one to make translation tool of CABRI-II like a way to drag an object with construction's constraint, like the orthogonality of the sides.

2) For the second session, we asked the pupils to characterize, using their algebraic knowledge, the families of lines SQ (Fig 1) which intersections with AF give the plot of the curve, the one the students had observed at the beginning of the first session. We then asked the following questions :

1. *Put an orthonormal landmark on the cabri-screen from which you have made your machine. Draw the straight line SQ. Display the coordinates of the points S and Q. By moving S on the vertical axis, we obtain a family of straight lines (SQ). Put S in different positions (6 for example) and write down on your sheet each time the coordinates of S*[8] *and Q corresponding to a each position. Then, construct the different straight lines by placing the points noted on the grill. Do the same work on the drawing sheet that has been given to you.*
2. *Do the straight lines you have drawn have something in common? and what? Name d1,d2,d3,d4,d5,d6 the six straight lines.*
3. *Choose a pair of points (S,Q) among the one you have noted and determine the slope of the corresponding straight line(SQ).*
4. *Redo 5 times the same thing with the other pairs of points*
   d2 slope:
   d3 slope:
   d4 slope:
   d5 slope:
   d6 slope:
   So, in your opinion, the straight lines d1, ..., d6 are a family (yes, no, why).
5. *We now take an interest in the straight lines (AF) when S moves along the vertical axis. Are they also forming a straight line's family (yes, no; if yes: why).*
6. *Use your modelling of Descartes' machine to plot the curve with locus tool of CABRI II.*
   Place 3 points on that curve and display their coordinates. Check with calculation that these points are not aligned.

Pupils have been very interested during this second session of the PW too, and it turned out to be a good pinpointing context for revisiting the them of the slope of the straight line.

3) Finally, the third session of the first PW, the one that aimed to find the equation of Descartes' hyperbola, turned out to be the toughest. We had top give precise indications of where to apply Thales theorem. See for example, the following part of the PW that pupils have worked:

4. *Now, it is about applying Thales to the triangles ASN and this one formed by the vertex S and the measure's segment x, which we are going to name MP.*

a) *Obtain a cabri-construction like the one as follows:*

b) *Write here a proportionality relation between the perpendicular sides of triangles ASN and MSP indicated.*
The relation found is: ________________

c) *Knowing that the straight line's slope determined by SQ is fixed and known, give him the value 1. You have to find an expression for the segment MS in terms of x and of this slope's value.*
MS= ______________

5. *Knowing that the points A and F are also fixed points, known points, we are going to place them as follows:*
- place the point A at a distance of 1 from point S
- place the point F on (1;0)

6. *Because the value of AS =1, we have to express the segment AM in terms of this value 1 and of x.*

AM= _____________

7. *Now, write here a proportionality relation between the indicated perpendicular sides of the triangles AOF and AMP. The relation found is:*
.....

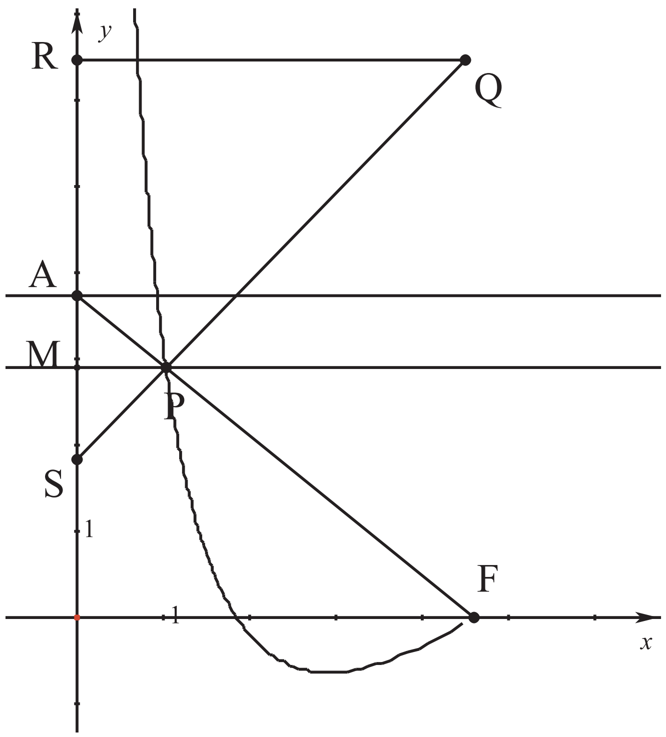

Fig. 2

Several pupils were close to obtain the equation of the curve. During the timed session, approximately 50 mn, despite its interest and the planned guidance, the students have only been able to establish the proportionality relations asked for. It was outside the class, later, that we asked advanced pupils to finish that task. After reworking ten minutes, they wrote the equation in question: $(f(1;x) - 2 + x = y$. We have completed this teaching experience with another second PW of two sessions of one hour each, about the graphical representation of basical functions at CABRI-II, in relationship with solving the algebraic inequalities (like $f((x-1)^2;x) = f(x^2;x+1)$ and others basical inequalities for this academic level).

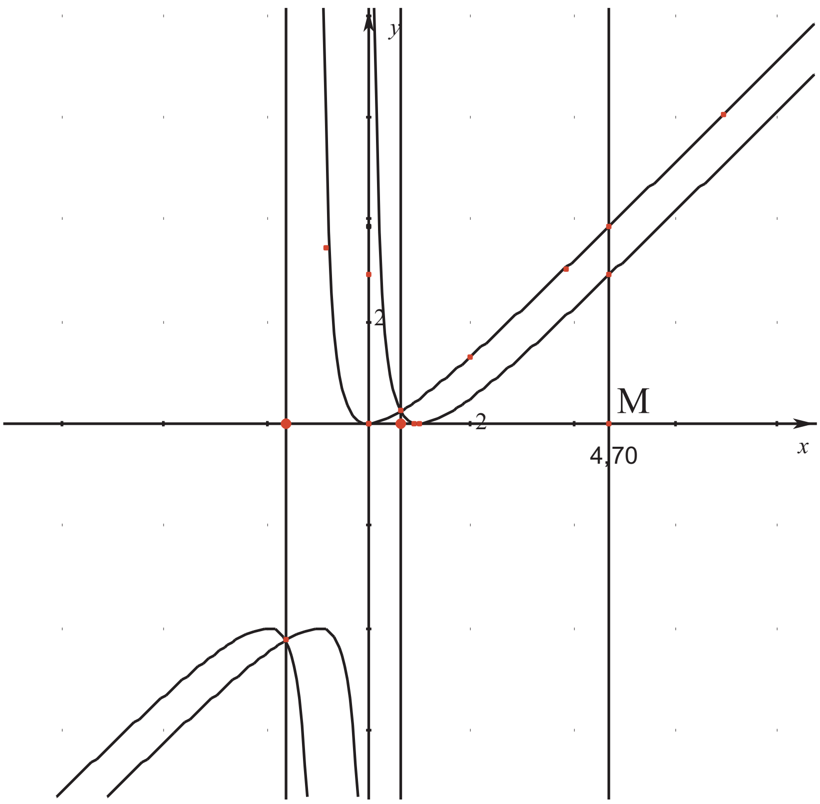

Fig. 3

This kind of inequation can’t be solved by mean of table process because when solving you need the study of the sign of $- x^2 - x + 1$. At this level students have not tools to solve it. So they need to use a graphical method with a graphic calculator or with Cabri-geomètre II.

First, we have noticed that most of the students were using the strategy of the crossed product, or that the table sign's change, to solve them. In this first case the pupils would solve without being preoccupied by the corresponding signs. In the second case the pupils couldn't find the correct solution. Already, at the totality of the second PW, we ask the pupils for the graphical pinpointing in CABRI-II of the curves denoted by each member of the inequalities. Finally, we have got one hour session of « echos » production, specially for reassess the connexions established about the geometrico-algebraic procedures studied, by argumenting with the pupils about their solution's procedure for the inequalities; inequalities similar to these they have already been confronted with (in the assessment sessions). They will also be able to evaluate their disposal tools worked in CABRI-II.

We think that this revision of resolution's own procedures and construction of controlled tools give the pupils the occasion to validate their usual algebraic manipulation as long as an analytical solution to the kind of exercises demands the plot of hyperbolas, straight lines, and parabolas; graphical drawings which plot and equation will be called by the introduced "voices".

In conclusion, the work hypothesis that we assume here is that a structuring of work sequences like these we presented would allow the pupils to establish links between the graphical drawing of the curves and their corresponding algebraic expressions; connexions that could more particularly provide the mathematical experience developed in[9] CABRI-II.

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Notes

1. In Hoyos' observation (1996) in the task of making the graphical drawing of the straight line 2x + y = 16; the students start to reduce the given equation and obtain correctly y = 16 - 2x. But they had not managed the calculations from the reduced equations:

   PB: To give values to x: look, if I give a value to x , two; it is going to be 4, plus , plus y, equals 16 minus 4, it means that y is 12.

   Thus, one of the cases (PB) indicated the equation 2x+y=16 while he was calculating out loud and after he had indicated y = 16-2x, finally he wrote y=12.

   ...

   PB: But I need another point...so here I put 4; multiplied by 2 is 8; 2 multiplied by...8, plus y equals 16, 16...

   When x is 4, y is 8. And here I already have 2 points to draw the straight line.

   (PB case interview, annexe 5 in Hoyos, 1996).

   We have to note that, this time again, the student PB was calculating by putting up with the form 2x+y=16 and not from y=16-2x.
2. Windows' Version of Cabri II. (This version is provided by Texas instruments).
3. Lycée Aristide Bergès Seyssinet-Pariset 38170. France
4. By the way, we have made a study (Hoyos1996) about the emergence of algebraic equations in mathematics' history. We think that Descartes with his Geometry (Gillies, D. (Ed.), 1992), Revolutions in Mathematics, Clarenda Press, Oxford science publications (USA)) has accomplished one of the mathematical resolutions, the one that represents algebraically some curves named geometrically. According to Descartes, at the beginning of the second book of "La Geometrie", it is possible that the reason why ancient geometers could not go further in the study of curves more complex than the conic ones was that the first considered ones where "the spiral, the quadratic and similar which belong to nothing else but mechanics" (Descartes 1637, p317).

   In "La Geometrie", Descartes shows us another way of considering the curves; a new way completely revolutionary for the time:

   « Et il n'est besoin de rien supposer pour tracer toutes les lignes courbes, que je prétends ici d'introduire, sinon que deux ou plusieurs lignes puissent être menées l'une par l'autre, et que leurs intersections en marquent d'autres... prenant comme on fait pour Géométrique ce qui est précis et éxact, et pour Méchanique ce qui ne l'est pas; et considerant la Géométrie comme une science, qui enseigne generalement à connaitre les mesures de tous les corps, on ne doit pas plutôt exclure les lignes plus composées que les plus simples, pourvu qu'on les puisse imaginer être décrites par un mouvement continue, ou par plusieurs qui s'entresuivent et don't les derniers soient entièrement reglés par ceux qui les precedent. Car par ce moyen on peut toujours avoir une connaissance exacte de leur measure. » (Descartes, 1637, p.316)

...

« Je pourrais mettre ici plusieurs autres moyens pour tracer et concevoir des lignes courbes, qui seraient de plus en plus composées par degrées à l'infini; mais pour comprendre ensemble toutes celles, qui sont dans la nature, et las distinguer par ordre en certaines genres, je ne sache rien de meilleur que de dire que tous les points, de celles qu'on peut nommer Géométriques, c'est à dire qui tombent tous quelque mesure précise et exacte, ont nécessairement quelque rapport à tous les points d'une ligne droit, qui peut être exprimé par quelque équation, en tous par une même. » (Descartes, 1637, p.319)

5. Ministère de l'Education nationale, de l'Enseignement Superieur, de la Recherche et de l'Insertion Professionnelle. (1995). Mathématiques -classe de seconde, classes de premières et terminales , séries ES, L, S-, France: Centre National de Documentation Pédagogique.
6. According to Boero.et al.(1997), « un 'echo' (it's) a link with the voice made explicite through a discourse »; &, a voice (it's) « some expression (that) represent in a dense and communicative way important leaps in the evolution of mathematics and science ». In the theoretical framing worked by Bartolini, M.(1995) & Boero,P.et al.(1997) they suppose that the introduction in the classroom of the voices of the mathematical and science hystory « might (by means of suitable tasks) develop into a voices and echoes game..» (Boero et al., 1997, p.81).
7. EIAH (Environnements Informatiques pour l'Apprentissage Humain), is a working team of Laboratoire Leibniz at IMAG & Université Joseph Fourier, in Grenoble, France.
8. Cabri-géomètre II gives coordinates of each point.
9. Briefly explain the importance we give to work on the CABRI-II microworld, we quote the words of Noss, R.&Hoyles, C., (1997, pp.132): « the system carries with

it elements of what is to be appreciated, it arranges activity so as to web solutions strategies. The structures which form that web of meanings are, of course, determined by pedagogical means: they are not arbitrary. There is human intervention, mathematical labour, dormant within the system, and the learner has to breathe life into in. »

THE INTERWEAVING OF ARITHMETIC AND ALGEBRA: SOME QUESTIONS ABOUT SYNTACTIC AND STRUCTURAL ASPECTS AND THEIR TEACHING AND LEARNING

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Abstract: *After a brief introduction and a few general considerations on syntactic difficulties in the interweaving of arithmetic and algebra, we analyse the conflict between additive and multiplicative notation in arithmetic-algebraic realm and present the first results of some activities carried out according to our hypothesis of research with the aim of overcoming such conflict and promoting the semantic control of complex writings. We conclude with some reflections concerning the choice and difficulties of the didactic activities in middle school on these topics.*
Keywords: *algebra teaching, learning syntactical, structural aspects.*

1. Introduction

There are several studies devoted to the problems connected to the passage from arithmetic to algebra, and specifically to the pupils' difficulties; scholars have different opinions on the actual possibility of overcoming such difficulties through appropriate teaching. Herscovics (1989) states that not all the difficulties that the pupils have on passing from arithmetic to algebra are due to the kind of teaching, since many of its obstacles derive from the way this passage developed historically. Harper (1987) too, believes that the pupils' learning repeats the historical process and therefore meets obstacles and difficulties witnessed by the history of the developing of the algebraic thinking.

Nevertheless, many studies show that the difficulties in the approach to algebra are often caused by a teaching of arithmetic focussing on the results of calculus processes rather than on its relational/structural aspects (see the surveys by Kieran 1989, 1990, 1992 or by Malara 1997a, 1997b). The teaching of elementary algebra stresses on symbolic representations (expressions, equations, functions) created by generalizing the processes enacted by the mathematization of various situations (Sfard 1991, 1994). This shifting from the process to the object is not sufficiently underlined in the teaching and the unspoken changing of perspective gives the pupils great confusion and often makes them accept passively the rules and techniques of formal calculus without any control of the meaning conveyed by these rules or of the properties on which these techniques are based.

The work that we present is included in one of our research projects (see Malara 1994, Malara et al. 1998), aimed at constructing *with and for* the teachers an innovative didactic itinerary in middle school (pupils aged 11 to 14) that on the basis of the research results and according to Italian syllabuses could allow a less traumatic and more aware passage from arithmetic to algebra. The project aims at overcoming the old Italian teaching tradition which introduces algebra *ex abrupto* in third grade and deals with it only from the syntactical point of view. It promotes learning algebra as a language (either in its syntactical aspects or in translation and of production/communication of thinking) *through and for* the study of problems, even those internal to mathematics and demonstrative ones. (see Malara &Gherpelli 1994 and 1997). Through the project we try to study if and to what extent a constructive teaching and in particular an early introduction of letters in parallel with a constant work of reflection and control of the meanings of which they are bearers- may limit or even overcome well known obstacles and difficulties.

The work focuses on some important questions related to syntactical, relational and structural aspects of arithmetic and algebra. More precisely, it concerns:

- the relational network (to be revised) between arithmetic and algebra in ‘elementary’ teaching;
- some aspects of arithmetic that can cause particular difficulties either within the learning of arithmetic itself or of algebra;
- the introduction of classroom activities concentrating on the additive and multiplicative notation as well as multiplicative/exponential (and their

combinations), whose difficulties should be highlighted by meaningful protocols.

It is not easy to write these questions in order, because are tightly intermingled. However, we shall face them starting from the cultural and didactic problems they bring about, presenting the activities we created for their potential overcoming, or at least for focussing on the difficulties detected. The activities are still being experimented in various classes and so have not been included in the project yet.

2. A Few Considerations on Syntactic Difficulties in the Learning of Algebra

Moving from arithmetic to algebra definitely creates didactic and learning problems: indeed you may see the algebraic code as a generalization of arithmetical symbols; still, as didactic research has highlighted, there are a few differences between symbolism in arithmetical realm and symbolism in algebraic realm. One *reason for this fracture is the multiplicity of meanings or roles that the same symbol (having a univocal meaning in arithmetic) obtains within the algebraic language*. See *parentheses* for instance: in the language of arithmetic they are used only to indicate the priority of an operation over the others when this priority contrasts with the convention; in the algebraic language the symbol ( ) can play the same role, but can also be used as a mere barrier between two signs you may not write one beside the other. This new function of parentheses often brings the pupils to the mistake of using parentheses in their first function only: this way they get to mistakes such as: $(-2+5)x(-4)=+3x-4$ or $(-2+5)x(-4)=+3-4$, which changes completely the meaning of the operation required.

On the other hand, you need even better control in expliciting or not expliciting some operation signs as soon as you start omitting completely the sign of multiplication between two literal symbols. In numerical realm the insertion of the dot instead of the sign x in general is accepted, but the fall of this tiny operational sign gives vent to ambiguities and confusion. Even in a rather short and simple writing like 2ab, sometimes the pupils insert the additive notation interpreting it as 2a+b. And should they face even more complex writings, they would have much bigger difficulties in managing all signs at the same time, be they explicit or unspoken, referring to numbers or operations. Sometimes syntactic errors are not due to the lack of understanding of the various situations considered one by one. As it generally happens, the simultaneous management of a multiplicity of situations determines errors that would never occur individually (Drouhard & Sackur 1997).

Even the signs '+' and '-' give vent to errors of interpretation: in the algebraic language they don't stand for operations only (addition and subtraction); they can be part of the number itself, the relative number; the sign '-' can be used as a unary operator to indicate the passage to the inverse. This last option causes further problems if you want to lead the pupils to a good awareness in the use of symbols: since they are instinctively induced to consider numbers as natural numbers, they hardly accept that a number can be expressed by a couple of signs, namely a sign and a natural number, and belong to an additive structure in which each element has its opposite. They also find it hard to accept the idea of the inverse of a given number in the use of rational numbers. Some protocols show that there are pupils who describe the following algebraic formula: $-(a^2b+5b)$ as 'the difference' of the product of the square of a and b added to the quintuple of b, which means they don't notice that in the subtraction there are two terms and here there are not.

A typical example of the various meanings of the signs is given by $\left(+\frac{1}{2}-3+\frac{1}{5}\right)\text{x}(-5)$ where a parenthesis is only a separator the other one has a double role, a sign '-' represents a unary operator, the other one is part of the number.

Moreover, as underlined by Demby (1997) and Reggiani (1996), it must be said that weaker pupils who can control the meaning of signs and conventions hardly yield to formal simplification and generally get back to the additive model, as shown in the following examples (reported by Demby): i) the expression $-2x^2+8-8x-4x^2$ is transformed into $-2x^2+8+(-8x)+(-4x^2)$; ii) the expression $(-4x+3)-(-1+2x)$ is transformed into $(-4x+3)+(-1)(-1+2x)$ or even into $(-4x+3)+(1-2x)$.

Another thing that should be considered, as underlined by Booker (1987), is the fact that, despite decomposition into factors, multiplicative representations of numbers are rarely used to carry out operations in arithmetic, whereas this kind of representations are often used in algebra as soon as letters are introduced. The operation of power on which such representation is based, gives remarkable problems in arithmetical activities and confirms its difficulties when operating on literal symbols. The impression we have is that the presence of letters emphasizes these obstacles to the extent of making it impossible to go any further, which is also due to the fact that the absence of numbers does not allow to sidestep the application of the properties of powers by resorting to 'naive' calculations of numerical kind. See, for instance, the different strategies (both wrong) enacted by the pupils in the following two situations: $(2^2x3^2)^3=(8x9)=72$; $(a^2xb^2)^3 = a^3xb^6$. Moreover, in the case in which a multiplicative writing introduces a factor with unitary exponent, errors appear also in the numerical case. The expression $(2x3^2)^3$ is transformed into $2x3^6$. It seems that the pupils multiply the exponents only if there is one of them already explicit for the basis; the number lacking in exponent is neglected and writes once again without expliciting the exponent. Yet, in the numerical case, many overcome such kind of situations by developing the product and the power within the parentheses and avoid to face the power of power. In the literal case this is impossible and the mentioned error occurs in grater percentage.

A further difference regards the sign of equality. Kieran (1990) underlines the different meaning that it has in arithmetic and algebra: in arithmetical realm pupils usually consider it to be a directional operator with the meaning of "gives vent to" (which is proved by the reluctance with which they watch expressions like 4+3=6+1); in algebra instead, even if this meaning appears in the simplification of expressions, the meaning of relation of equivalence prevails and it achieves significance in the study of equations (Herscovics & Linchevski 1992 and 1994).

This difference in dealing with literal and numerical expressions, with the consequent errors, appears evident when you work on the two levels at the same time. When the pupils, for example, were asked to operate simultaneously on the two expressions $(2x3^2)^3+(2^2x5)$ and $(ab^2)^3+(a^2b)$ by using the properties of powers as much as possible, 50% of the class avoided the obstacle as follows: in the arithmetical case they calculated the product that is at the basis of the power first and obtained the right result; whereas in the literal case only 10% of the pupils operated on powers by elevating each factor of the product within the parentheses to the more external exponent, all the others just got stuck in front of the exercise.

3. The Confusion Between Additive and Multiplicative Notation: A Hypothesis of Inquiry for its Overcoming

We believe that for most of the syntactic errors detected it is possible to sort out a few basic reasons that allow a unitary vision of the problems that the pupils have. Among them, the main reason, which is also at the basis of our hypothesis of research, is the following: the additive structure of the naturals and the consequent additive notation constitute a very strong primitive mental model for the pupils, particularly in primary school: the multiplicative model, which is introduced right afterwards, is not as strong and often it doesn't overlap correctly to the additive: it is less spontaneous, even because it is learned later. Didactic research offers many examples about the prevalence of the additive model in problem solving. Still, here we are talking about the prevalence of the operation of addition as to multiplication not only semantically, but also from a formal point of view. The pupils focus their attention on the sign '+' and this leads them to see '+' in the situations in which the operative sign is not explicit (as we said, sometimes the simple writing 2ab is interpreted as 2a+b). Moreover, as emphasized by Fishbein (see Fishbein & Barach 1993 or Fishbein 1994), owing to their formal elegance and visual incisiveness, some additive laws, such as m(a+b)=ma+mb, overcome their status of laws and become strong models for the pupils, who extend them inappropriately in multiplicative realm, confusing the roles of addend and factor.

The additive notation dominates and gets confused with the multiplicative one in many situations. For example during a first grade activity in which the pupils were asked to do transformations in order to calculate mentally more quickly, some pupils transformed 148x20 into 148x10+148x10. This testifies that they could more easily see 20 as 10+10 rather than as 10x2. When asked to find the inverse of the rational number 3/5, some pupils declared they had been thinking of another fraction like 1-3/5, which means they confused the inverse fraction with the complementary one. However, the field in which these difficulties are more evident is where you operate with powers and their properties. The product or quotient of powers with the same basis requires the simultaneous management of both operations and notations: the multiplicative level between powers turns into the additive one between the exponents, to which occurs the difficulties due to the management of the power as a binary operation. Unlike the other arithmetical operations that the pupils know ever since primary school, it is an operation in which the sign is not explicit; moreover, the role of the two elements of the couple is not the same: the exponent is just the index of how many times you have to write the same factor (the basis) in a repeated multiplication. Even if the various calculating techniques are learned by the pupils in a preliminary phase, the complexity of an expression that simultaneously contains more operations of the same kind raises lack of control over signs and meanings. The question can be expressed as follows: *if the conflict that the pupils experience between additive and multiplicative notation can cause errors or confusion in the learning of the algebraic language and of its meanings, is it didactically useful- and to what extent - to find out situations and strategies to force the comparison and exchange between additive-multiplicative situations and multiplicative-exponential situations?*

The research we started with second- and third-grade pupils (teacher: R. Iaderosa) moves right from this assumption and aims at analysing the pupils' ability to spot out (by intuition) the operative "structure" transferring it from the additive field to the multiplicative one, and backwards.

Comparison between the multiplicative and additive realms

Middle school, second grade (pupils aged 12-13).

1) By using any time it is possible the properties of powers, calculate:
   a) $3^3 \times 3^2 + 3^3 + 3^2$; b) $5 \times 2 + 5^2$; c) $(2 \times 3^2)^2 + (2 + 3^2) \times 2$.
2) Transform the following writings by replacing each sign of addition with a sign of multiplication, and each sign of multiplication with a power:
   a) $(2 \times 3) + 5 + 7 \times 2 \rightarrow$; b) $(2 \times 5) \times 7 + 2 \rightarrow$; c) $(5 + 2) \times 4 \rightarrow$.
3) Transform the following writings by replacing each sign of multiplication with an addition and each power with a multiplication:
   a) $2^3 \times 5 \times 7^2 \rightarrow$; b) $5^3 \times 2^4 \times 3 \rightarrow$; c) $(5 \times 8) \times 2 \rightarrow$; d) $(5^2 \times 2^2) \times 3 \rightarrow$.
4) Look at the following writings and say whether each of them is true or false:
   a) $2 \times 5 + 3 \times 4 = 5 \times 2 + 4 \times 3$; b) $2^5 \times 3^4 = 5^2 \times 4^3$.

   Look at a) and b) once again. Is it possible to pass from one to the other like in the previous exercises?

Middle school, third grade (pupils aged 13-14).
By using as much as possible the properties of powers, calculate:
a) $2^3 \times 2^2 + 2^3 + 2^2$ $a^3 \times a^2 + a^3 + a^2$
b) $(2 \times 3)^2 + (2^3)^2$ $(ab)^2 + (a^3)^2$
c) $(2 \times 3^2) + (2^2 \times 5)^2$ $(ab^2) + (a^2b)^2$

Tab. I

The activities, purposely arranged, have the aim of forcing the comparison between the multiplicative and additive notations by opposing them in complex situations in combination with the analogous multiplicative and exponential notations, with the aim of improving the distinction between the two realms and spotting out analogies and differences among their properties. The first productions by the pupils showed that 50% of them could keep good control over the two notations. As foreseen, they had difficulties in operating with powers and some of the weaker pupils simply tried to explicit powers as products. In particular, while they understand pretty well the correspondences (direct and inverse) between addition and multiplication, they cannot control the operation of power at the same time, especially in the inverse passage, as shown by the following examples of solution of the exercises in table 1: i) exercise 2a) (2x3)+5+7x2 was transformed into: $(2^3)5\times7^2$; $2^3+5\times7^2$; $2^3\times5\times7^2$; 2x3x5x7x2; ii) exercise 3a): $2^3\times5\times7^2$ was transformed into: 2^3+5+7^2 ; (2x3)x5x(7x2); (2+3)+5+(7x2). However, after a few months of activity on these aspects, the pupils showed better ability in grasping the differences between the two notations and a wider control of even more complex situations. Some interesting protocols on these activities will be showed during the presentation.

4. Considerations and Problematization of Some Arithmetical-algebraic Activities in Middle School

As a matter of fact, algebra - seen as study of relations and procedures - begins and must begin from the study of arithmetic. Still, to what extent the knowledge of arithmetic can help when shifting to algebra? The relations between the two fields are much more complex. For example, a deeper knowledge of arithmetic, of the formal properties of the arithmetical operations, of the various numerical realms can favour the correct learning of literal calculus and of some concepts tied to equations, algebraic operators, etc. Nevertheless, it is the algebraic code itself that allows the comprehension of higher level arithmetic. For example, the generalization of a rule or procedure that would hardly be understood through a single listing of numerical cases can be expressed in a literal code. The same way, some arithmetical properties are deduced from a literal formalization. Paradoxically, in some cases the algebraic language can simplify the analysis of arithmetical contents: it is well known that many of the properties concerning the relation of divisibility between integers or the structure of some natural numbers as to their factorization escape the numerical form and are highlighted by a literal writing.

Particularly refined aspects of the teaching of arithmetic are implicit in problems concerning the calculus with powers. In fact, if we consider the power at integer exponent, defined in R, and the multiple after an integer of a real number, we can grasp more easily the analogy between the two definitions and how in both situations we may not speak of commutativity, since the two elements of the ordered couple are not seen as belonging to the same domain. Unfortunately, at this level of schooling, when pupils only just approached such topics and therefore do not operate on R but on N, it is objectively very difficult for the teachers to make them distinguish the two levels. It is therefore clear that the operation of power itself (usually introduced in first grades) for many reasons can be hardly and definitely 'unnatural' to first-grade pupils who by no means possess such algebraic tools of analysis. As to this, one could try to highlight the analogy between the following two writings: 2×3 and 2^3. By formally representing the two writings as: $(2,3) \rightarrow 2+2+2$ (where 2 is written 3 times); $(2,3) \rightarrow 2\times2\times2$ (where 2 is written 3 times)

It is evident that the analogy consists in the fact that in both cases only the first element of the couple is involved in the calculation, whereas the second expresses only the number of times the addend or the factor is written.

So, it is not a matter of 'doing' algebra only *after* having introduced arithmetic, but rather of reading and developing didactically the algebraic aspects of arithmetic, which shouldn't necessarily be delayed to the 'end' of the arithmetical contents. This should be born in mind especially for the middle school, when teachers and pupils must not pursue the technical skills that allow to deal new contents at superior level, and therefore can find the occasions for reflecting systematically on these questions. Indeed, often it is the ripening of the pupils' thinking that allows them to overcome some cognitive obstacles; still, there are concepts upon which they have never reflected enough, even when they get to university.

The introduction of letters focuses on the aspects of arithmetic that the pupils hardly ever interiorize: first of all the operation of power and the recognition of its properties, the distributive property, the concept of inverse and opposite of a number, seen as numbers first, and then as operators. And in fact it is from these situations that the syntactical problems arise, since the limited and recent arithmetical experiences have not allowed the appropriate interiorization. What has been learned is not an automatism yet, whereas in this field one should, to quote Bell (1992), "*make what has been learned automatic and vice versa recognize in the automatism its meaning*". Let us consider situations like the following: $(2\times3)^2+2(2\times3)$ and $(ab)2+2ab$. In the numerical case, almost inevitably, in the pupils' minds the development brings to the form 6^2+2. In the literal form, operating with powers brings to the form a^2b^2+2ab, which explicitly suggests the application of the distributive property and leads to the expression $ab\times(ab + 2)$. The reading of the two formulas obtained is definitely different, and this should make the pupils reflect ion the fact that the letters highlight relations and forms that the numerical aspect somehow hides. This confirms the need of operating on the possible equivalent expressions of a same number or of a numerical expression, as emphasized in Malara (1994).

On the other hand, it is also necessary to consider the following points:

- how can a formal property for an arithmetical operation be fully understood if the pupils still cannot abstract from single numerical example an equality among procedures and express it in a literal form?
- how can they recognize an inverse element as to the multiplication in numerical realm, if they still find it hard to consider fractions to be numbers?

On gradually introducing the use of letters instead of numbers, we saw that the same paths and properties that seemed to be known, often are not recognized as such if they appear with a new, or different, symbolism. This is evident as regards the recognition and application of the formal properties of the arithmetical operations. Let us consider the following example in which we report the various steps, in order of difficulty, that should be explicited and understood in numerical realm in order to get to recognize and apply correctly the distributive property in transforming 2ab+b into (2a+1)b: i) 3x2+5x2=(3+5)x2; ii) 3xa+5xa=(3+5)xa; iii) 2xa+ax3=(2+3)xa; iv) 2xa+axb=(2+b)xa; v) 2xa+a=(2+1)xa; vi) 2ab+b = (2a+1)xb. Protocols on the difficulties in facing these steps will be showed in the presentation.

Thus, our belief is that only by carrying out a metacognitive teaching of arithmetic, introducing early and gradually the use of letters and dealing from the very beginning with the algebraic aspects which are contained in arithmetic itself is it possible to guide the pupils towards the achievement of a correct use of the algebraic code, and in the meantime start their acquisition of the relational and structural aspects which in future perspective shall bring them to possess, at the end of high school, the tools to comprehend the study of abstract algebra.

Finally, a special mention goes, as already underlined by Gallo (1994), to the problem of control. Probably this problem has a different relevance and connotation for middle-school pupils from older ones. Very often, the lack of control is not due to a separation from the meanings, but more simply to the lack of autonomy in the application of what has been learned. Many pupils find themselves in the ‘zone of proximal development’ and need to be guided along the reproduction of an itinerary that was learned in school, they simply cannot operate correctly in an autonomous way. Moreover, when this control exists, if it works in an acceptable way in situations in which the difficulties signalled by the literature are ‘diluted’ in a wider numerical or literal realm, it turns out to be inadequate in situations in which you have to manage more than one ‘crucial’ question at a time. But we believe that presenting these questions systematically in complexity can emerge a real control, unrelated to the acquisition of mere (though correct) automatisms: an almost automatic awareness.

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SYNTACTICAL AND SEMANTIC ASPECTS IN SOLVING EQUATIONS: A STUDY WITH 14 YEAR OLD PUPILS

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Abstract: *The capacity to write and to solve equations is a crucial point in the construction of algebraic thinking, since it involves both the mastery of the formal rules of algebraic language and calculation, as well as the correct interpretation of the meaning of the symbols used. Our research study is mainly concentrated on examining the possibility of improving the capacity of students to solve first grade equations during their first year of high school, by establishing a dialectical relationship between the application of properties and their semantic control. In this report we would like to present certain observations concerned with the problem exposed in relation to equations without solutions or with an infinite number of solutions.*
Keywords*: symbols, rules, meanings.*

1. Introduction

Research into teaching and learning algebra has demonstrated that one of the fundamental problems is the students' difficulty in being able to manage a formula and its meaning at the same time. (Arzarello et alii 1995, Malara 1997, Sfard 1991, Cortes 1993, Linchevski & Herscovics 1994,…)

When considering in particular the capacity to write and solve equations, numerous research studies have brought to light various other aspects which are included in this process. In particular, studies have been carried out concerning certain problems linked with using the equal sign in its relational sense (Sfard 1994) and the various uses of letters as unknown quantities and as parameters (Ursini 1997). One of the fundamental aspects which cannot be ignored when working with equations, is the fact that while the equation in general represents the symbolic translation of a problematic situation, and the solution of the equation leads to the solution of the problem, the various intermediate passages cannot always be easily read as the stenographic transcription of the resolving strategy (Boero 1992). It should also be remembered that numerous studies have demonstrated the fact that students of 11 to 12 years of age, and sometimes even primary school pupils, are already capable of successfully applying linguistic type or ”elementary” method strategies (inverse operators , graphs, etc...) to solve equations which are linked (or not) to a problem situation (Reggiani 1994, Cortes 1993).

Our research involves the study environment which in classical terms could be defined as the syntax - semantic relationship, with special interest dedicated to the problem of semantic control over the operational methods (Gallo 1994). This work also takes into consideration the result of research carried out previously by our group on problems connected with the passage from arithmetic over to algebra - and especially with reference to the problems of the generalising and the use of symbols and conventions in algebraic language. Our observations have shown that alternating the use of different environments (arithmetic, computer information, geometry) for an approach to algebra seems to be an efficient method for making semantic control of algebraic operations easier.

2. Research Context

From the results observed after certain secondary school entrance tests, we focussed our attention on some of the answers given by pupils to questions which demanded the solution to simple equations, directly or in the form of inverted formula. An analysis of the results has shown performance levels much lower than work carried out by the same students in other parts of the same test. According to literature, we observed that in these cases, the aspect which was lacking was the control of the results and of the transformations carried out.

So it was decided to study the possibility of improving the pupils’ capacity during their first year at secondary school (14 year olds) as far as solving equations was concerned, improving the semantic control both with respect to the context in which the problem is posed (for example in the case of inversion of formulas, or of equations used to solve a problem) as well as with respect to the meaning of the algebraic operations made by the pupils themselves.

In fact, we feel that the solution to an equation, which requires good competence in applying the formal rules, represents the synthesis of the passage from arithmetic to algebra only if a process of generalisation (abstraction) has been achieved which refers to the meaning of the objects of the algebraic language, and if the pupils have understood the semantic implications of the formal rules and the operative conventions of algebraic language.

Often, as shown also by some of our previous research studies, these capacities have not been acquired and this results from the fact that the pupils try to replace them with ”rules” learnt in a mechanical way. Through a large number of examples introduced at teaching level (with the use of diagrams, verbal explanations, class discussions, etc.) our research concentrates on identifying the point of equilibrium between operating methods and the meaning of the operations that the pupils are working on - that is: between syntactical and semantic aspects - while being well aware of the difficulty of improving the learning capacity on both levels: in fact, as several authors have pointed out, very often the excessive attention to meaning slows down the necessary training in self control needed for the pure application of the rules for manipulating expressions (automatism).

3. Research Problem

In the environment of the problems described in the previous paragraph we are studying the persistence of the fracture between operations and meaning in algebraic contexts where formalism (linked to the capacity for abstraction) becomes more important, in particular when dealing with equations without solutions, with infinite solutions, or with ”unacceptable” solutions.

4. Research Method

The work phase described in this report was carried out in the following manner: in a first year secondary school class composed of 27 pupils, during normal scholastic hours, a teaching program of 15 hours was set up between the teacher and the research group on the subject of first grade equations and based on the observations which had emerged during previous research by our group (Bovio et al. 1995). Also present during the lesson was a student in her last year of a Mathematics degree as an observer.

At the end of the program a set of written and individual tests was carried out in order to check the level of competence the pupils had acquired in relation to first grade equations, and in particular, in situations where it was necessary to keep in mind the meaning of the operations being worked on. The analyses provided in this report refer to the individual protocols of the pupils.

5. Summary of the Work Carried out in Class

Since the discussion concerning the protocols under examination cannot be separated from the activities carried out beforehand, a short summary of the whole learning program is essential.

The work in class was concentrated on the following points:

- Recovery of the pupils' ability to solve problems through equation using elementary methods in order to establish a connection between the work proposed and the activities carried out by the pupils previously (generally during intermediate school). Through observation of spontaneous strategies this stage was directed at attributing meaning to equation solving procedures. In fact, these procedures often compose the generalisation for elementary methods of solution (the use of inverse operators for example, or using the example of balancing scales, or the simple ”verbal solution” often used in the simplest cases), but often the pupils do not establish any connection between procedures and spontaneous strategies.

- Solutions, using elementary methods or through attempts, of equations even on levels higher than first grade equations which are easily factored or are not polynomial (for example exponential), with the aim of generalising the meaning of the term equation and to clarify the meaning of the term solution.
- Solution to first grade equations through operations of transformation by trying to underline the meaning of the transformation by emphasising the necessity that transformations are inverted. During this stage, we also used tables in order to provide a kind of graphical diagram to the solution of the equations through transformations, and at each passage, emphasising the set of the solutions and comparing that set with the one identified previously. Obviously during this stage we spent some time on the problem of the ”zero”, comparing its different roles when it is an additive or a multiplicative term or a denominator. We discussed with the pupils the fact that equations can have a finite number of solutions, (underlining in particular the cases where there is only one solution) or can have no solution, or an infinite number of solutions.
- Discussions of the problem of the presence of letters which are different from the unknown in an equation, and how to deal with denominators.

In agreement with the idea of learning proposed by the theories based on constructivism the work methods in class were based on the individual study of problems and then on the following group discussion on the solutions proposed by the pupils.

6. Verification Tests

The tests run at the end of the program were composed of various verification checks which included two problems expressed verbally to be converted into an equation and (subsequently) solved; the solving of certain first grade equations of various levels of complexity, one with letter coefficients; and certain equations not belonging to the first grade, to be solved using elementary methods.

Here we would like to focus a particular problem:

”Consider the equations of each of the following groups and reply whether they have the same solutions, giving reasons for your answers:

a)	2x-7 = 5	2x = 5+7	2x = 5-7	x-7/2 = 5	x-7/2 = 5/2
b)	0·x = 5	x = 5	x = 0	x = 0/5	0·x = 10
c)	x = -3	0·x = -3·0	x+2x = -3+2x	x·x = -3x	
d)	(x+2)/3 = (x-5)/2	2x+2 = x-5	x = x-7	0·x = -7 “	

7. Analysis of the Proposed Problem

The four groups of equations represent different situations. In each group there are equivalent equations and non-equivalent equations. The equations are generally obtained from each other by applying the usual transformations (adding or subtracting, multiplying or dividing both members by the same quantity); however in some cases we reproduced typical mistakes, obtaining non equivalent equations.

As we mentioned before, the pupils from the previous teaching program have worked on these transformations identifying the limits of application; repeated use was also aimed at establishing them as rules. It is necessary that the solution of a first grade equation becomes (at least in part) one of the automatisms necessary to be able to deal with more complex problems.

The object of the exercise therefore is to verify the pupils’ capacity to identify the cases where it is not natural (or is incorrect) to use the rules they have learned. Based on the kind of didactic activity previously carried out, we feel that when pupils make this choice they should be guided by the meaning of the operation they are working on. In particular, in the class discussions, we insisted heavily on the meaning of ”resolving an equation in a set” - understood as : ”How to find the values which replace the unknown in order to make the equality true”; during the discussion we also underlined that only the transformations of an equation which maintain that it is true for the same values, without adding others, may be accepted.

Of the four groups of equations proposed:

The equations of group a) checked only the capacity to apply correctly transformations normally used to solve a first grade equation with numerical coefficients where the unknown appears only once.

The equations in b) checked the capacity to recognise first grade equations without solutions, to see that in this case the fact that the equations have ”the same solution” does not depend on the fact that they are obtainable from each other using the normal transforming methods. Alongside impossible equations, we included simple equations which correspond with possible incorrect attempts to apply the usual rules. This group of equations focalized attention on the role played by ”zero” with a certain amount of emphasis.

The equations in c) reproposed the problem of multiplying by zero, checking the pupils’ capacity to carry out cancelling operations in relation to terms including the unknown .

The equations in d) lastly, proposed the problem of eliminating the denominator in a simple case and faced the problem of equations without solutions in the case where ”x is cancelled”.

Each group of equations proposed different situations, and therefore it was not used to check whether the pupils overcame a single difficulty. Since the check was at the end of a program where we worked separately on the single aspects examined in this test, we felt it was necessary to propose them in a more complex way, creating a new situation, where the pupil was confronted with a context that did not reproduce the teaching examples he had experienced previously in class, - in other words - exactly what happens when one has to apply acquired knowledge in a different context.

However, these tests were compiled in such a way to permit pupils and teachers to examine the various aspects singly.

8. Analysis of Results

Out of the 27 pupils who sat the test, only 8 solved the problem correctly and completely, providing adequate reasons for their answers; 14 pupils obtained intermediate results, indicating that on the whole, they were able to carry out the transformations which conserve the solutions of an equation; on the other hand the remaining 5 pupils showed serious problems.

In the next paragraph we will analyse the performance of the five pupils with reference to their protocols.

As far as the other students are concerned, without providing a complete picture of the results for each single item, we have limited the analysis by underlining the fact that all the pupils answered point (a) correctly; almost all compare correctly the first equation with the third and the fourth in point (c), and the first two in point (d).

Pupils had most difficulty with the equations in point (b) and also in the other exercises where they had to establish the equivalence between equations without solutions, or with infinite solutions.

Among the group classed as ”intermediate”, we noted that certain pupils do not carry out a complete analysis of those points, or else - they give correct answers but without any explanation; or else they make a mistake when resolving one comparison, but carry out others which are just as difficult, correctly. Therefore we feel that in spite of the fact that they tend to apply the ”rules”, even when solving problems where attention must be paid to the ”meaning”, these pupils manage to keep situations even more difficult than usual from a semantic point of view, under control.

Even keeping in mind the fact that certain aspects are incomplete, or lack explanations, we can conclude that these pupils have perhaps acquired ”more complete rules”.

9. Analysis of Single Protocols

In this paragraph, we will analyse the protocols of the pupils who showed the greatest difficulty. We will limit this analysis to those points described in the previous paragraph as most important.

Pupil 1

(point b): "*Taking 0·x=5 I multiply both members by 0 and obtain 0·x=5·0 - that is x=0 which is equivalent to 0·x=5. But 0·x=5 is not equivalent to x=5 because the results are different. x=0/5 is not equivalent to 0·x=5 because 0·x=5 is an identity while 0·x=10 is equivalent to 0·x=5 because when both are multiplied by zero, both are equivalent to x=0.*"

For this pupil, all the equations of point (c) are equivalent because they are obtained from one another by multiplying (or adding) both members by the same quantity. In relation to point (d) the same pupil states that "*in the equation x=x-7, if you subtract x from both members you obtain 0=-7 and not 0·x=-7*".

In the protocol of this pupil we notice the presence of mistakes in syntax such as stating that 0·x=5·0 is equal to x=0, that is, that 0·x is not 0 but x. As is easily noted, this conviction is persistent. It is also obvious that the pupil tends to apply rules and tries to use terms which she has not yet understood the meaning (0·x=5 is an identity). We also note the use (positive in itself) of different explanations for the various situations but without operating any control over the coherence of the results and the explanations. (0·x=5 is equivalent to x=0 because I can obtain both equations from one another; it is not equivalent to x=5 because the results are different: for this pupil, what is the result of 0·x=5, which as we noted, was first defined as an identity?)

Pupil 2

(point b) "*0·x=5 is equivalent to x=5/0 which is also equivalent to x=0. But x=0/5 cannot be equivalent to x=0 and therefore to the equation assigned initially*"

(point c) "*From 0·x=-3·0 I obtain x= -3·0/0, that is x=-3*"

In the same way the pupil explains that "*from x·x= -3x I obtain x=-3*"

As for point (d) the pupil limits her observations to the non equivalence of the first two equations "*because the denominator is not the same*".

The pupil deals with all the equations by following the layout where the syntax solution of the equation ax=b is x=b/a . In the first exercise there is also a syntax error which we found frequently in the entrance tests: that is a/0=0 . The pupil also states that on the contrary 0/a is not 0. In the second exercise the pupil simplifies 0/0=1.

Pupil 3
(point b): ”*x=0 is equivalent to 0·x=5*” (without explanations). No other answers are given for this point.
All the equations of point c) are equivalent”
”*The three first equations of point d) are equivalent*”.

Pupil 4
(point b):”*x=0 and x=0/5 are not equivalent to 0·x=5 because only the second member is zero*” The pupil made no comments on 0·x=10. In the same way for point (c) the pupil establishes the equivalence between the other equations but does not take the second into consideration without supplying any explanation.
In the same way in point (d) the pupil does not comment on the last example.
Pupils 3 and 4 probably suffer from the same difficulties as those already observed, however the fact that they supply no explanations shows a behaviour pattern which is perhaps even more confused when dealing with the cases where it would be necessary to verify if it is possible to use the methods they have learnt.

Pupil 5
(point c): ”*By multiplying by zero a false equality can be made true, so, by moving from the first to the second equation, the equality has become true*”
Concerning other points - the pupil repeated the same mistakes already met previously.
This pupil supplies an interesting explanation: ”*The multiplication by zero makes a false equality true*”. However all forms of conclusion are missing as well as any type of more precise connection with the case in hand. It would seem that the pupil repeats even at explanatory level a statement which probably came up in class discussion during a lesson but that he did not understand the exact meaning and limited himself to learning it as another ”rule”.

10. Final Remarks

The presented test mainly gave positive results since it demonstrated that for most of the pupils the learning process was activated: a teaching program, like one we have described, concentrated on an ”operation-meaning” relationship, seems to have led the pupils to applying the rules using a ”reasoning process” which permits most of them to use the rules correctly in most cases.

However there are certain cases where a lack of success was very evident: the pupils who are not capable of the effective learning process make an effort to apply rules but are not able to exercise any control over the passages they do; this becomes clear especially in the presence of zero, when it is essential that the pupil has a good understanding of the meaning.

These pupils are also hindered by their difficulties with arithmetic - such as confusion between multiplying and dividing by zero, or the fact that they are not aware that 0a=0. Very often they try to repeat definitions or explanations at the very moment when they are not able to attribute a precise semantic value.

This type of lack seems to be qualitative, in that it does not concern having or not having worked out a certain number of operations to consolidate the use of certain rules, but rather the inability to include them into a synthetic structure where each element obtains its own value through its interrelation with the others.

This was already the aim of the teaching program carried out, but evidently these pupils were not able to participate.

This does not mean that these pupils are excluded from the possibility of reaching the goals which have been envisaged, but we think that results such as these will be constructed (in their personal mental experience) not so much through the sedimentation of operational practice, but more probably, suddenly in the context of an adequately organised individual discussion based on exercises already carried out.

We feel that this type of evolution can be the object of a continuation of this research study.

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FORMING ALGEBRA UNDERSTANDING IN MPI-PROJECT

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Abstract: *Research in how to form Algebra understanding has been done in frames of MPI-project for many years. It is based on H.Weyl-I.Shafarevich conception about Algebra as the collection of coordinatizing quantities systems. There are three main informative lines in MPI-project systematic course of Algebra-functional, algebraic structures and mathematical modelling. Two last of them are discussed here.*
Keywords*: algebraic structures, modelling, computer visualization.*

1. Introduction

We present a part of the project of school mathematical education ”Mathematics. Psychology Intelligence.”(MPI-project, Tomsk, Russia), the head of the project is Prof. E. Gelfman. The project is directed at the development of students’ individual cognitive experience.

The problem of Algebra understanding implies two other problems: the nature of algebra and the nature of developing intelligence of a child. In short, teaching Algebra should be organized in such a way as to reflect the role of Algebra as the human intellectual culture phenomenon and, on the other hand, to provide the development of individual intellectual abilities of a child.

2. Weyl-Shafarevich Conception

What is Algebra? Out of numerous possible approaches to answer this question, I would like to attract your attention to H.Weyl - I.Shafarevich conception. H.Weyl : ”... now we are coming back to old Greek viewpoint, according to which every sphere of things requires its own numeric system defined on its own basis. And this happens not only in geometry but in new quantum physics: physical quantities, belonging to a certain given physical structure, permit themselves (but not those numeric values which they may assume due to its different states), in accordance with quantum physics, perform addition and noncommutative multiplication, forming by this some world of algebraic quantities, corresponding to this structure, the world, which cannot be regarded as fragment of the system of real numbers” [1].

I. Shafarevich summarized these ideas of H.Weyl in such a way :

a) every phenomenon, every process of real world (also in mathematics itself) may be ”coordinatized” in the frames of some system of coordinatizing quantities, that is represented in some generalized system of ”coordinate axes”;
b) subject of Algebra is a study of various systems of coordinatizing quantities as concrete (for example, numbers, polynomials, permutations, residue classes, matrices, functions and so on)as well as abstract (groups, rings, fields, vector spaces and so on);
c) if some phenomenon is not yet coordinatized by any familiar system of coordinatized quantities, the problem of coordinatization arises. The task of algebra is to solve this problem, that is to make up a new system of coordinatizing quantities. Its further study may or may not be connected with problem of coordinatization, with solving of which the given system of coordinatizing quantities appeared. [2]

3. Basic Elements of Algebraic Knowledge

This viewpoint turns Algebra from forum to explore the power of abstraction and formal logic to the collection of coordinatizing quantities systems which are meaningful for real world knowledge. Therefore we have to consider it necessary to include the elements of abstract algebra in school curriculum. The next question arises: how to form the understanding of Algebra by children? What requirements should a school text meet for it? As for the informative aspect the following basic elements of algebraic knowledge should be included in the school text organization:

1. Algebraic symbolics as the universal abstract language for describing reality.
2. Algebraic operation in the context of all its basic properties.
3. Algebraic structures as the specific form of coding information.
4. Algebraic notions semantics as the premise for the reality specific aspects realization, aspects connected not only with the sphere of usual human experience, but also with the sphere of human ”impossible” experience.

Thus, mastering algebra means mastering new language, new methods of knowledge, new forms of the information organization, new reality view.

However for algebra to be perceived by a child in these most important aspects, the psychological preparation of a child is necessary. It includes:

a) forming up conceptual experience gradually working with the subjective space of a child’s understanding of separate signs meaning as well signs expressions;
b) developing maximally his imagination experience for the generalization as the visualization ability allows to ”catch” the sense of mathematical notions without any verbal-logical argument;
c) complicating his object operating experience, including both the main mental operations and heuristic experiments and the mental experiment prediction;
d) forming the metacognitive experience of a child;
e) having the individual intuitive experience as the basis;
f) forming the multidimensional mental space for school material understanding.

4. Algebra Propaedeutics

Traditionally in Russia the systematic algebra course begins for children of 13. The practical school experience shows that in spite of traditional previous preparation, there exist some serious difficulties in the process of children’s understanding algebra. For example, a psychological barrier of ”unwillingness” to accept algebraic symbolics appears and children are not able to use algebraic language as the means of increasing effectiveness of their intellectual work. Moreover, children begin to be afraid of abstractedness of algebraic school material. We consider that many a difficulty is explained by the fact that the informative and psychological lines do not equally exist in algebra teaching. In consequence the cognitive experience of a child is not reconstructed enough for a child to be psychologically prepared for the specific character of algebraic material.

Gradual and prolonged introduction of letter signs in themes devoted to number sets taught to 10-12 year old children in MPI-project[1] has a definite aim: children must intelligently estimate the role of a separate letter and algebraic expressions as an intellectual instrument, making them able to fix and express relations between any number objects in a visible and compact way, to discover the general way of solving analogous tasks, to investigate hidden regularity in the sphere of this own subject and number experience.

Some ways of methodical work aimed at the introduction of a letter sign and letter expression in 5-6 grades (10-12 year old) lead to understanding that the central moment which influences the progress of a skill development on the propaedeutics step is the level of the formation of the number sign subjective image. The introduction of algebraic symbolics on the propaedeutics step of algebra course certainly can't be reduced to the mere understanding of an algebraic expression as a specific language for reality description by a child. It's necessary for a child to get a skill of using different problems. Thus a certain methodical problem emerges - to form a special style of thinking of pupils. We make them know the main elements of problem solving with the help of equations. They are taught to choose the variable, the basis for making up a equation, the information necessary for problem solving under conditions of expressiveness or insufficiency of data specially. We teach pupils to correlate the result of equation solving with the word problem conditions, according to this equation, to create different figurative notions of initial problem (in the form of drawing, schemes, tables etc).

The pupils are taught to understand that one and the same real situation can be expressed mathematically with the help of different equations and also that sometimes outwardly different problems can be solved with the help of one and the same equation.

What concerns the psychological aspect of activity of a child, the preparation to study algebra presupposes to form the skill to see regularity, to guess, to ask question, to make hypotheses, to prove them and to contradict them, to work in the frames of the pattern: how if it were...

5. Mathematical Modelling Line

The work using the algebraic symbolics as a special language for reality description and for forming children's special thinking style presents an opportunity to give various problems to the pupils requiring attaching different mathematical apparatus while organizing systematic algebra course. Here is, for example, one of the ecological problems proposed for pupils of the 9 th grade (15 year old) in the textbook " Equation systems".

Foxes and rabbits live on the island. There is enough grass for rabbits to eat and foxes eat only rabbits. It is required :

1) to find out how the amount of foxes and rabbits is to change at the initial moment of time so that at the given moment their amount is the preassigned number;
2) to predict how many foxes and rabbits will be after the definite time.

Pupils come to know the scheme of mathematical modelling, developing the scheme of its work while solving word problems. They get a skill of work with the obtained model, to make charts of relation of foxes and rabbits amount to the time and also a curve of interdependence of foxes and rabbits on the plane "foxes - rabbits". Then pupils use the modelling skills answering the question: ''What if...?" For example they find out the consequences of the island administration decision concerning the changes in foxes and rabbits number at the initial moment, learn which of these decisions lead to the destruction of the ecological system "foxes-rabbits". The increase of factors number from two up to three, four and so on lead pupils to the necessity to solve linear equation systems of corresponding order. Besides the mathematical modelling in this case must be confirmed by the computer modelling. Systematic work with pupils in this direction using ecological as well as economic models (salary - inflation, investments - consuming, demand - price - supply etc) helps to form up multidimensional dynamic thinking, which is so necessary for every school graduate in this complex quickly changing world.

6. Algebraic Structures Introduction

The systematic course of algebra in MPI-project includes three main informative lines - functional, algebraic structures and mathematical modelling[2]. Functional line is traditional and we don't dwell on it. Some description of mathematical modelling line has been done before. Now we consider algebraic structures line.

The extension of number systems takes place in the 5 - 6 grades. Pupils come to know fractions, decimal fractions, negative numbers. Every time they face the necessity to find out the nature of new numbers, to study operating with numbers. We think that at this step pupils can get the first experience of working with algebraic operation as an object: they can learn the procedure of operations introduction and their properties. One of the motives for new numbers introduction can serve the impracticality of a definite operation on the "old" number set. For example, the introduction of negative numbers occurs while studying the practicality of the subtraction operation on the natural numbers sets. Pupils should pay attention to the fact if the new introduced operations have the properties of commutativity, associativity, distributivity. Certainly it makes no sense to speak about any strict proof, but we have an opportunity to create the experience of checking the presence or absence of these properties.

The work with the number sets described above gives an opportunity later to develop the functional aspect as well as the algebraic structures line in the systematic algebra course. Realizing the functional aspect we used the pupils experience with the operations on numbers, involving them into the search of the definitions of the operations on monomials, polynomials, algebraic fractions. Developing the aspect of algebraic structures we can go on analysing operations and objects worked at. So, we want pupils to meet absolutely new for them operations. Some examples of such operations have been done in the first book of the algebra systematic course "Making Friends with Algebra" (13 year old).

Task 1. Are whether the addition and multiplication the algebraic operations on the set:
a) odd numbers; b) even numbers?

Task 2. Is whether the subtraction the algebraic operation on the set:
a) naturals; b) integers?

Task 3. Is whether the division the algebraic operation on the set:
a) nonzero naturals; b) nonzero rationals?

Task 4. Check that the operation

$$x (y = x + y - 3$$

is algebraic one on integers.
Is whether this operation associative and/or commutative?

Task 5. Check that the operation

$$x (y = x + y - xy$$

is algebraic one on integers.
Is whether this operation associative and/or commutative?

Task 6. Make up some own examples of operations over numbers. Which from them are algebraic one? Which from your algebraic operations are associative and/or commutative?

Such work with pupils allows children to get the experience of independent constructing of algebraic operations with definite properties. Thus psychologically full understanding of operations presupposes the children to use freely their knowledge about operations even under unusual ”non-task” conditions. Such creative attitude to an algebraic operation can be stimulated showing children the fact, that operations can be fulfilled not only on numeric but on non-numeric objects as well.

The fact is that earlier children used the number notion in such problems which required measurements. Moreover, every widening of a number notion was caused by the necessity to measure one or the other value, which couldn’t be measured according to the previous number notion. The same line is continued in senior grades, when the real and then the complex numbers are introduced. Finally the secondary school graduate has a stereotype formed. It is: ”everything can be measured by a number or a corresponding widening of a number notion”.

So as to destroy the stereotype mentioned above we make pupils know permutations and work at them. It is performed in the form of a game in the part ”For those who want to have secret correspondence with friends” of the book ”Making Friends with Algebra” in which the permutations are used to ”measure” the process of coding and decoding texts. For instance, let us take the word ”timer”. We denote all letters in this word by numbers of positions which they occupy in the word. If we want to code this word, then we choose any order of numbers from 1 to 5, for example, 2, 5, 1, 3, 4 and then we have the figure with two rows, which is named as permutation. This figure has the meaning: in upper row there is initial order of letters in the word (i.e. 1, 2, 3, 4, 5), in lower row there is the order of letters in coding word (i.e. in our case 2, 5, 1, 3, 4) and numbers from upper row transfer to corresponding numbers from lower row (i.e. in our case 1 transfers to 2, 2 transfers to 5 and so on). The permutation which gives the coding word is named a coder and as result we have the coding word ”mteri”.

If we think the coder insufficiently secures the secret information, we can repeat the same procedure with other coder to coding word and as result we have twice coding word. For instance, let us take the other coder as permutation which upper row is 1, 2, 3, 4, 5 and which lower row is 1, 5, 2, 3, 4. If we apply this permutation as coder to coding word ”mteri”, then we receive twice coding word ”merit”. What is the coder which transfers initial word ”timer” to twice coding word ”merit”? In the first coder we have 1 transfers to 2, in the second coder we have 2 transfers to 5, then as result 1 transfers to 5 and so on for 2, 3, 4, 5. Finally we have twice coder as permutation, which upper row is 1, 2, 3, 4, 5 and which lower row is 5, 4, 1, 2, 3 (i.e. 1 transfers to 5, 2 transfers to 4, 3 transfers to 1 and so on). The twice coder is considered as the result of product operation for two permutations.

So we are coming to definition of algebraic operation over permutations. Basic properties of this operation also have real world interpretation. For instance, the permutation with identical rows is named as identity permutation and corresponds to transferring information without coding. Let us give a permutation, which is a coder, and we change upper row with lower row in it, then resulting permutation performs decoding and is named as decoder. After applying the coder to initial word and then applying the decoder to coding word we receive the initial word. So the coder is inverse element to the decoder relative to algebraic operation of permutation product. Also such properties of this operation as associativeness and noncommutativeness are given.

At last pupils appear to be intellectually prepared to perceive the ideas of classification of algebraic and numeric structures.

We invite pupils to meet the planet ”Quarta” [3] under conditions of this fantastic planet children have to work with the number like objects in the situation when it is impossible to create the fractions field of some polynomial ring. Pupils compare what is common for numeric systems and number like objects. Finally they have an opportunity to go over to the notions about group, ring and field.

7. Computer Visualization of Algebraic Structures

So we come to the hypothesis : forming Algebra understanding passes through such stages as visual images- intuitive ideas-abstract notions- logical reasoning. This process needs computer support. It is given by computer program ”Visual Algebra”, which was worked out under my direction by Tomsk University computer science undergraduate A.Shprenger. The idea of computer visualization of algebraic structures is an application of discrete functions for constructing three-dimensional and two-dimensional images of a given algebraic structure. Moreover they can manipulate by these visual images, i. e. visual images are dynamic. Therefore the same visual images from ”Visual Algebra” may give different intuitive ideas to students depending on the state of their individual mental experience and psychological individual learner’s style. Then the transition from visual images to intuitive ideas is an interesting research area for a teacher. For instance pupils may set different algebraic operations independently and to explore their properties. It is possible to see two – and three-dimensional images of the assigned algebraic operations and to compare corresponding visual images.

This way is suitable for mixed-abilities classes. Every student realizes the transition from visual images to intuitive ideas by the way which is most suitable for him. The key problem – how this diversity of intuitive ideas will come to the correct abstract logical reasoning? It requires high level of competence from a teacher. The most typical form of intuitive idea is the recognition of visual images, for example, to recognize visually different properties of algebraic structures.

As an education example we regard two groups: cyclic group of order 4, which is looked upon as a subgroup of symmetric group of the fifth degree, generated by a cycle of length 4 and Klein's quarter group, regarded as a subgroup of symmetric group of the fifth degree, generated by two cycles of length 2. Comparison of visual images of these groups leads students to the hypothesis that these groups are not isomorphic. The next step is logic basing of this hypothesis by comparison of elements orders in these groups. At the same time the consideration of visual images for permutations cyclic group of order 4 and residue classes additive group by modulo 4 leads to the hypothesis that these groups are isomorphic. Logic basis of the given hypothesis may be found by construction of isomorphism among the pointed out groups as concrete map.

Finally I invite all researchers, educators and everyone else who are interested in ”Visual Algebra” contact me for cooperation in exploring the possibilities, which are given by ”Visual Algebra”. You are welcome to the beautiful world of visual algebra!

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Notes

1. The series of textbooks for children of 10-12 in MPI-project includes the following textbooks:
   i) “Decimal fractions in the Moominhouse”; ii) “Positive and Negative Numbers in Pinocchio's Theatre”; iii) “The Case of Divisibility and Other Stories”; iv) “About Helen the Beautiful and Ivan the Prince and Fractions”.

2. The systematic course of algebra includes the following textbooks: “Marking friends with Algebra”, “Identities”, “Algebraic fractions”, “The Book about the Roots from which New Numbers Grow”, “Quadratic Equations”, “Inequalities”, “Equation Systems”, “A Fairy-Tale about the Sleeping Beauty or a Function”, “Quadratic Function”, “Sequences”.
3. A part “Ksysha on Quarta” of the book “Algebraic fractions” is devoted to this problem (8 grade ,14 years old).

SOME TOOLS TO COMPARE STUDENTS' PERFORMANCE AND INTERPRET THEIR DIFFICULTIES IN ALGEBRAIC TASKS

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Abstract: *The aim of this paper is to show how some research tools coming from different disciplines and theoretical frameworks were used to explore students' difficulties in a rather complex algebra task and the differences between two different groups of students, in order to produce interpretative hypotheses about emerging phenomena.*
Keywords: *tools, comparison, performances, algebra.*

1. Introduction

A research study was performed by us at the Teacher Training College of ELTE (Budapest) about the use of a set of tools (derived from different sources) to analyse students' performances and interpret their difficulties in algebra tasks. These tools concerned both a-priori and a-posteriori analysis, and included peculiar keys to analyse processes (anticipation, exploration, cognitive unity between conjecturing and proving, etc.). In this paper we will exploit some of these tools (which will be described in the a-priori analysis section - see 2.2. and 2.3.) to analyse and compare Italian and Hungarian university students' performances in the following task: "*Generalize the property: "The sum of two consecutive odd numbers is divisible by four", and prove the generalized property*". The aim of the reported study was to produce interpretative hypotheses about emerging phenomena.

This report is based on the protocols of 33 Italian and 41 Hungarian students who worked on the task for approximately 90 minutes. The Italian protocols were made by three groups of 4th year mathematics students at the Genoa University. After completing their 3-year course on mathematics, they chose to specialize on mathematics teaching. The Hungarian protocols came from two groups of 2nd year students, one group of 3rd year students and one group of 2nd year evening-course students. All of them are trained for teaching mathematics (for children age 10-14) from the beginning of their studies at our college. All of them had already finished one semester on number theory before the task was given to them (no course of this kind was attended by Italian students). The task was quite unusual for both the Italian and the Hungarian students, it seemed to be appropriate for collecting data about the students' ability in making generalizations, conjectures and proofs.

2. A Priori Analysis of the Task

2.1 Some Possible Solutions

There are several ways for generalizing the original statement.

- We expected most the following:
  ”The sum of 2m consecutive odd numbers is divisible by 4m.
  Different proofs are possible, for example:
  > Proof 1: *The sum of 2m consecutive odd numbers is:*
  > S = 2k–2m+1 + 2k–2m+3 + 2k–2m+5 + ...+ 2k+2m–5+ 2k+2m–3 + 2k+2m–1
  > S = 2m · *2k = 4mk ; finally we get: 4m |S.*
  >
  > Proof 2: *Starting with 2k+1, the nth odd number is 2k+2n–1. The sum of n consecutive odd numbers is S = 2k+1 + 2k+3 + 2k+5 + ... + 2k+2n–1.*
  > Applying the formula for the sum of an arithmetical sequence we get:
  > S = n(2k+1+2k+2n–1)/2 = n(2k+n). If n is even, e.g. n = 2m,
  > then S = 2m(2k+2m) = 4m(k+m), which is clearly divisible by 4m.
- There are some ”weaker” generalizations as well, for example:
  ”The sum of an even number of consecutive odd numbers is divisible by 4.”
- There are also some other ways of generalization (see 3.2. C).

2.2 Reference Knowledge

i) The task needs some basic knowledge of arithmetics and algebra like:

- concept of divisibility, consecutive odd/even numbers. These concepts are obviously necessary to understand the task;

- algebraic representation of divisibility and of consecutive odd/even numbers;
- ability to express the n^{th} term of certain series of numbers;
- appropriate use of parameters.

These skills refer to the ability and confidence at the algebraic level. They are basically necessary if someone wants to give a proof for a meaningful general statement.

ii) Mastery of some formulae, results in elementary number theory and methods ready-to-use as a tools, are necessary or may be very useful:

- transformation of algebraic expressions;
- the formula for the sum of the first n odd or at least even numbers or the general formula of the sum of an arithmetic progression or the ability to construct or re-construct such a formula quickly;
- induction method.

Transformations of algebraic expressions are necessary in proving. The knowledge of a useful formula can be a great help at the proving stage but if missing, it can be replaced (for example by constructing and proving the formula). Induction can be useful at that case.

iii) The necessary metaknowledge about:

- generalization: some understanding of the meaning of generalization is strictly necessary at least on an intuitive level;
- the structure of the statement of a theorem (hypotheses, thesis). It is needed to write down the statement in an appropriate way and prepare the proving stage;
- proof: the knowledge of the requirements needed for mathematical proof (as a deductive chain, etc.) is necessary not only to write down the final product, but also to construct the proof.

2.3 Processes

Producing the conjecture and arguments supporting it and constructing the proof needs:

A) Anticipation

It is a key process, which enters both at the semantic level (planning appropriate numerical experiments) and at the algebraic formalism level (planning appropriate transformations). (see Piaget, 1967; Boero, 1997).

B) Exploration
(see Polya,1962; 1973; Simon, 1996; Boero et al., 1996).

- Checking the original statement (through arithmetical examples or by algebraic formalization);
- looking for variables to be left free in order to find a way for generalization;
- arithmetic or algebraic exploration of the situation:
  - looking for numerical regularities
  - exploring general formulae in order to select a regularity expressed in algebraic terms
  - testing the idea of a conjecture in specific cases

These are constitutive elements of the complex processes explicitly demanded by the task.

C) Producing the statement
- Elaborating ideas (guesses for a stable statement, arguments supporting it) taken from experiments and explorations;
- Constructing and expressing the conjecture in words and/or by formula.

D) Proving
- Detecting and exploring appropriate arguments from the preceding phases: formalization of arguments, interpretation of algebraic expressions (Arzarello et al., 1995), possibly establishing a ”cognitive unity” with the production of the conjecture (see Mariotti et al., 1997; Garuti et al., 1998);
- producing new arguments, if necessary. It may require transition from one frame (e.g. arithmetics) to another (e.g. algebra) (see Arzarello et al., 1995);
- connecting the appropriate arguments in a deductive chain.

The passage from algebraic formalism to the semantics and vice-versa (”interpretation” and ”formalization”) are key elements in proving and in some cases in the creation of the conjecture.

2.4 Difficulties

- Difficulties depending on the subject’s capacities and knowledge may arise from lack of the anticipatory thinking and/or lack of the adequate arguments.
- Difficulties depending on specific aspects of the strategy may arise from:
  - breaking the continuity between producing the conjecture and constructing the proof (see Garuti et al., 1998; Mariotti et al., 1997) (this breaking may concern the kind of exploration, formalism, frames and reference culture);
  - lack of flexible usage of formalism (it means: adapting the formalism to the situation) (see Duval, 1995).
- Difficulties with ”metaconcepts” (about generalization, proof, etc.)
- Difficulties related to the didactical contract (Brousseau, 1986): under an unusual task like this, students try to guess the teacher’s expectations deriving from their past experiences, but those experiences are far from the present task, so their effort may be oriented in an inappropriate direction.

2.5 A Priori Expectations

Our expectations were based both on the preceding a-priori analysis and our knowledge about students’ background and performances in other algebraic tasks.

We expected the majority of the students to start by proving the original statement; than explore the problem arithmetically and/or algebraically, by increasing the number of terms, proving some specific cases and a minority to try to expand the problem in other ways. We expected that the notion of generalization would have not been entirely clear for the students, that their conjectures would have been a mixture of proper generalizations (weaker or stronger) of the original statement and of other statements somehow connected with the initial one. We expected that a good percentage of the students would have been able to prove the produced conjecture.

3. Analysis of Performances

3.1 How the Students Exploited or not Reference Knowledge

i) Every student was able to understand the original statement. We found that none of the students had problems with arithmetic elements like divisibility, odd or even consecutive numbers, and all of them were able to find an algebraic representation for two (or a few more) consecutive odd/even numbers. More problems emerged in the use of formalism, especially as mastery of parameters is concerned (to accomplish less standard tasks).

ii) The success of finding a solution depended strongly on the confidence in the field of algebra, e.g. on the ability to find a formula for the general term.

Some students had difficulties with the transformation of a formula when they tried to use the ? sign. Some students tried to use induction for proving divisibility (it may depend on a lack of anticipation at the usage of this method). Those who were familiar with the sum of the first n odd or even numbers were in a much easier situation then those who were not. In some cases, the use of induction could have been a help at that stage.

Being able or not to interpret an algebraic formula was a very important factor of success in finding and proving a meaningful conjecture; in many cases failure or success depended on it.

iii) There was a lot of confusion about the meaning of *generalization*. As we will show later, among the conjectures produced there were quite a few trivial generalizations and there were a lot which were true (or false) statements connected with the initial problem but not generalizations of that.

There were great differences at the success in *proving* the conjectures, depending on metaknowledge about proof. Indeed, while there were quite a few protocols in which we could not find any serious attempt for proving, there were others who became stuck at the beginning because of confusion about the nature of proof. There were also some students showing consciousness about some aspects of proof (generality, exploitation of hypotheses ...) but unable to detect a missing link in the deductive chain.

iv) The *didactical contract* seemed to have a significant influence on the products especially in connection with the generalization process and proving process.

3.2 Processes

A) Anticipation

Anticipation is an important factor both in the exploration and the proving phases. We will refer to its role there.

B) Exploration

Almost all of the students started their work with proving the original statement. Most of them constructed an algebraic proof but they were not equally confident in the algebraic representation of two consecutive numbers and of the sum of them. Some of them used the most proper parameters and representations quite easy to interpret, like:"$2k-1 + 2k+1 = 4k$" or "$2k+1 + 2k+3 = 4k+4$". Some others used more complicated systems of parameters, representations with more then one step, sometimes not covering all the cases, like:

"$a + a+2 = 2(a+1)$ *where a is an odd number" or*
"$4k+1 +2m+1 = 2(2k+m+1)$ *where m is an odd number*".

After (or without) proving the original statement almost every student started – consciously or instinctively, systematically or at random – to look for an explicit or implicit parameter which can be changed. All of them made – either arithmetical or algebraic – experiments with statements of the form "*The sum of any number of something is divisible by something*". They tried several possibilities.

Most of the students started to experiment with:

- the sum of 3, 4, ... consecutive odd numbers (instead of 2)
- the sum of any two odd numbers (instead of consecutive)
- the sum of two consecutive whole or even numbers (instead of odd).

Some of them changed more than one parameter of the original statement, for example experimented the sum of more than two consecutive whole numbers.

Some of them experimented with more hidden parameters, for example with $(2k+1)^n + (2k+3)^n$.

The students either tried to find cases where their sum is divisible by 4, or tried to change 4 for an other divisor.

The exploration of two consecutive whole or even numbers does not lead to a generalization. Those who chose this direction– either because of the lack of anticipation or because being very systematic while working on the concept of generalization– got stuck or changed for an other direction.

Many of those who insisted on the divisibility by 4 were not able to find more than weak generalizations. On the other hand, those who changed too many parameters at the same time during their experiments easily became lost while collecting lots of statements similar to the original one, or got (sometimes quite meaningful) general statements which were not generalizations of the original one.

As we could see, anticipation played a very important role at choosing a proper parameter to change.

Experimenting with 3, 4, 5, 6, ... consecutive odd numbers turned out to be the most promising direction. During the exploration the students worked on three different level of abstraction:

I. Level of number examples.

II. Level of statements we can obtain from the original one by changing one given value in it for another value. For example statements like ”The sum of three consecutive odd numbers is divisible by 3”. These statements have the same number of variables as the original statement and all of them can be fulfilled by infinitely many number examples. The students could get them either from arithmetic or algebraic experiments but their verification can not be done by using number examples only.

III. Level of statements we can obtain from the original one by changing a given value(s) for variable(s). For example statements like ”The sum of 2^k consecutive odd numbers is divisible by 2^{k+1} ”. A statement of this level was a common generalization of a subset of second level statements. The task was to find a statement of this sort where the subset contains the original statement.

The results of the students strongly depended on the level they are familiar with and on their ability of moving from one level to an other.

Those students who had very poor algebraic tools were confident only at the first level. Those of them who experimented only with number examples were able with a few exceptions to construct a good (in some cases even a strong) conjecture expressed in words but they had no tools for proving. Their success in finding a good conjecture depended on their ability of finding some order in the experiments.

In spite of being uncertain of the algebraic representation even of consecutive odd numbers, some of them tried to explore the situation algebraically. They often got stuck at the different parametrizations of either the original or an other second level statement.

Those students who were able to move on to work algebraically with second level statements mostly chose sooner or later to experiment with the sum of 3, 4, 5, ... consecutive odd numbers. They realized quite soon that the sum of 3 (5, 7,...) consecutive odd numbers is not divisible by 4, and they reacted to this fact in different ways. Some of them became confused, changed the direction by chance without anticipation, or changed the divisor too, or changed too many parameters. Some of them tried to construct a statement which is true for all the cases, like ”The sum of n consecutive odd numbers is divisible by n” or ”The sum of n consecutive odd numbers is divisible by their mean”. They forgot that the result should not be merely the common generalization of their second level statements but of the original statement as well. Many of them noticed that separating the cases (odd number of members and even number of members) would help and they got a good conjecture. Those again, who rigidly insisted on the divisibility by 4, got only the weak generalization.

C) Producing the statement

Some of the students did not give any statement as a generalization or considered a second algebraic representation of the original statement as a generalization. For example: "*Not only* $2k+1 + 2k+3 = 4(k+1)$ *is divisible by* 4*, but if p is odd, then* $2k+p + 2k+p+2 = 2(2k+p+1)$ *is also divisible by 4*."

Some of them gave only statements of the second level, like "T*he sum of three consecutive odd number is divisible by* 3." It is not always clear from the protocols if they considered these statements as generalizations or just as results of experiments.

Some of them gave third level statements which were generalizations for their second level experiences but not for the original statement. For example: "The sum of n consecutive odd numbers is divisible by n." Again, it is not always clear from the protocols if they considered these statements as generalizations or just as results of experiments.

Some of them gave a proper generalization but only a trivial one, like "*If two odd numbers have different remainders by* 4*, then their sum is divisible by* 4."

Many of them constructed meaningful generalizations like:

- "*The sum of* $2n$ *consecutive odd numbers is divisible by* $2n+1$."
- "*The sum of* $2k$ *consecutive odd numbers is divisible by* $4k$."
- "*If a and b are consecutive odd numbers and n is odd, then an+bn is divisible by* 4."
- "*If we take* n^2 *consecutive numbers and then leave out those which are divisible by n, then the sum of the remaining* $n2-n$ *numbers is divisible by n2*."

Algebraic tools were helpful but not necessary for constructing even a meaningful conjecture. Those students who were not successful at getting one either lacked anticipation and were not flexible when their experiences contradicted to their expectations, or were not able to keep a systematic order while making experiments, or were too uncertain about the concept of generalization.

Many of the students gave more than one (not always proper) generalizations. The quality and the amount of their statements also depends on what they thought the expectations were

D) Proving

While exploration was possible on the first two levels, to prove it is unavoidable to move to the third level. Now we deal only with those who arrived at a meaningful conjecture. The proving phase was strongly interconnected with the exploration phase in the students' works. There were great differences between the students in terms of the arguments collected before they arrived at the proving stage. The two extremes were:

- those who created a conjecture in words based only on numerical exploration;
- those who produced a correct closed formula of the sum of an even number of consecutive odd numbers, and their conjecture was only the interpretation of that formula.

With a few exceptions the students who tried to prove a general statement dealt with statements about the sum of consecutive odd numbers. The proof needs the ability to formalize the sum of n terms of a series and to give a formula for the general term (i.e. the ability to formalize their conjecture). To cope with these difficulties, the arguments one collected so far came very useful ("cognitive unity", see Mariotti et al., 1997). There were basically two ways of using the arguments collected during the exploration:

- The arguments were used immediately when they appeared, as a part of a continuous process. They served in these cases like the steps of the stairs for their reasoning.
- The arguments already left behind were taken out again when the progress met an obstacle or when there was a gap in the deductive chain. They served in these cases either as starting points to avoid the obstacles or element to bridge the gap.

These ways resulted in a continuity in all the processes. Breaking this continuity was quite rare, but caused disaster in most of the cases.

To go on with the proving process one had to be able to find a "closed" formula for the sum and to interpret it.

Finding a closed formula was very difficult for many students.

Some of them tried to use the Σ sign, but it did not help. The formal usage of algebraic symbols (without the clear understanding of their meaning) broke the continuity in many cases causing confusion rather than helping the students as the following example shows.

”$(2n+1) + (2n-1) = 4n$ *OK!*
$(2n+1) + (2n-1) + (2n+3) + (2n-3) = 8n$ *still divisible by 4, but by 8 as well!!*
Similarly, for the sum of consecutive numbers, we will always get $2n + ... + 2n$ this will always be divisible by 4, if I take even number of consecutives. Moreover if I take $2k$ consecutives, their sum is divisible by $4k$.
I prove what I have found: $(2n+1) + (2n-1) + ... + (2n+p) + (2n-p) =$
$S_{p=1 \text{ to } k}\, (2n+p) + (2n-p) = S_{p=1 \text{ to } k}\, 4n = 4n$ *but it can not be $4n$!!*”
(At this point he starts again without any more results.)

Some others tried proving by induction but it was evidently hopeless in this situation. These cases clearly show the lack of anticipation in the application of these techniques. Some students transformed the sum into another, perhaps a simpler but not a closed one. For example:

”*I translate the fact of having m (even) consecutive odd numbers:*
$2n+1, 2n+3, ..., 2n+(2m-1)$. Their sum is : $2nm + (1+3+...+(2m-1))$.
$2nm$ is divisible by $2m$, so it is enough to show that $1 + 3 + ... + (2m-1)$ is divisible by $2m$.
I know that the sum of two consecutive odd numbers is divisible by 4:

$\underline{1+3} + \underline{5+7} + ... + \underline{(2m-3)+(2m-1)} =$
$4 \bullet s \quad\quad 4 \bullet t \quad\quad 4 \bullet k$
$4 \bullet 1 \quad\quad 4 \bullet 2 \quad\quad 4 \bullet (m-l)$
$= 4(1 + 3 + ...+ (m-1))$ where the sum has m/2 members.”
(At this point he thinks that the last sum has odd number of members and tries to prove –by a hopeless induction– that the sum is divisible by the number of members).

To find a closed formula was a new ”problem in the problem” and they could start a new exploration-hypothesis-proof cycle (induction could have been helpful for proving the validity of a formula for the sum). This path could have been successful but no one could complete it in the given time.

Producing a complete proof demanded good anticipation, enough arguments from the exploration, some flexibility, confidence in the necessary algebra, and clear understanding of what proving means.

4. Comparison Between Hungarian and Italian Students

4.1 Findings

We have found many similarities between the Italian and Hungarian protocols, all types of difficulties and typical reasoning occurred in both cases. Now we will concentrate on the differences.

The differences we found fall into three main types: understanding the task; the use of algebra; consciousness about proving.

The Italian students were more aware of the concept of generalization. Some of them explicitly stated what generalization means (while none of the Hungarians) and quite systematically tried to find a proper free variable. Even if some of them could not produce a meaningful general statement, none of them moved away from the task. The Hungarian students had only an intuitive concept of generalization and many of them produced various general statements which were not necessarily generalizations of the original one. Some of them moved away quite far from the task.

Italian students had more problems with the necessary algebra and basic results of elementary number theory than the Hungarian ones. Some of them tried to use algebra on a level which they were not familiar with. The Hungarian students seemed to be more comfortable with using algebra and basic results of elementary number theory as tools.

Italian students felt more obliged to prove. Even if they were not able to complete a proof they were more keen on trying it. Some of the Hungarian students were more successful at proving (due to their algebra and number theory knowledge) but some of them did not even try to prove, and some were satisfied with a convincing argument.

4.2 Possible Interpretations

Although taken from small samples occasionally collected and not representative of Hungarian and Italian students, the preceding quantitative and qualitative data show the existence of important differences between the two groups of students probably depending on their pre-university and university background. Some of the research tools considered in this paper can provide us with interpretative hypotheses about phenomena emerging from our exploratory study. Further, more extensive and statistically significant comparisons should be performed in order to test these hypotheses, keeping into account that the complexity of the task and of the students' background involved in it represents a tremendous challenge for a serious comparative study.

Above the obvious cultural/educational differences we feel that the main reasons behind the findings above are the different curricula and the different didactical contracts. While in Hungary all the secondary school students learn and have to solve a lot of exercises about the sum of progressions and traditionally there is a great emphasis on number theory, we think that in Italy students are trained to be more conscious about the most important metaconcepts of mathematics. The produced protocols helped us to realize the importance of the didactical contract and to reveal some elements of it. There were significant differences even between the Hungarian groups taught by different lecturers. Hungarian students are not expected and have very little practice in reflecting to (and controlling) their thinking. They would not be able to put down private comments about their reasoning and difficulties even if they were asked for it. The fact that many Hungarian students considered the task as an open investigation problem was partly due to their previous education, but, as the differences between the Hungarian groups show, to the personal didactical contracts too.

5. Conclusion

The performed study confirms the need for tools of different sources (epistemology, psychology, didactics) in order to analyse this kind of task.

Epistemology and psychology tools seem to be sufficient to analyse individual behaviours; but when comparison between different groups is performed, the didactical contract (typical tool from didactics of mathematics) is needed in order to interpret differences which seem to be inaccessible to other interpretations.

Finally we would like to point out the importance of students' metaknowledge (about generalizing and proving) in open tasks like that considered in this paper.

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GROUP 7:
RESEARCH PARADIGMS AND METHODOLOGIES AND THEIR RELATIONSHIP TO QUESTIONS IN MATHEMATICS EDUCATION

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RESEARCH PARADIGMS AND METHODOLOGIES AND THEIR RELATIONSHIP TO QUESTIONS IN MATHEMATICAL EDUCATION

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– SUMMARY AND DISCUSSION –

(Elaborated by Christine Shiu)

The seven papers which comprised the starting point for our discussion and are published above seemed to fall naturally into three types: two dealt with global theories of mathematics education (Rouchier, Godino & Batanero), three were empirical studies which addressed specific questions in mathematics education (Bagni, D'Amore & Maier, Stein) and the remaining two presented accounts of elaborated methodologies which have been devised and developed for specific purposes in research into mathematics education (Marí, Stehlíková).

It was agreed in the group that we would allow one session of 45-60 minutes in length (known as an A-type session) for each paper to be discussed intensively and that such sessions would be interspersed with longer ones (90 minutes, known as B-type sessions) in which general issues emerging from the particular cases would be examined. In A-type sessions the author-presenter gave a brief synthesis of the paper and added any complementary information which was deemed appropriate. The chair then moved the group from questions of clarification into the identification of the issues arising.

The pattern of the sessions was as follows:

Day 1

Session A1 André Rouchier, chaired by Klaus Hasemann
Session A2 Juan Godino, chaired by Leo Rogers
Session A3 Giorgio Bagni, chaired by Martin Stein
Session B1 general discussion, chaired by Hermann Maier

Day 2

Session A4 Hermann Maier, chaired by Jeremy Kilpatrick
Session A5 Martin Stein, chaired by André Rouchier
Session A6 José Marí, chaired by Nada Stehlíková
Session B2 general discussion, chaired by Juan Godino

Day 3

Session A7 Nada Stehlíková, chaired by Gunnar Gjone
Session B3 general discussion, chaired by Christine Shiu

The resultant general discussion involved us in a consideration of the nature, roles and functions of theories in mathematics education and how these related to specific questions in mathematics education, and how both theories and specific questions shaped and were shaped by the methods used to investigate the questions. What emerged was a triad of theory, research questions and methods, which when developed and used in a particular context, could viewed as a research paradigm. It was noticeable that specific research paradigms seemed often to be particular to the cultural or national context in which they were developed.

It was soon apparent that interpretations of the meanings of many words which we were using differed among the members of the group. These differences too might be particular to their cultural or national context. Elucidation of differences framed the discussion as they were explored through a series of questions. The questions themselves were in their turn scrutinised and refined as the sessions progressed, and the following summarises the progress and scope of the discussion.

1. Theories in and of Mathematics Education

1.1 What is a Theory and How Do Theories Arise?

We took as a working definition that a theory is a "framework of concepts which show how things work".

Theories in and of mathematics education can be developed within mathematics education but often, quite properly borrow from and particularise theories developed in other disciplines such as psychology, sociology, anthropology, semiotics etc. Theories can therefore be imported from other fields of knowledge (and used unchanged or adapted to a certain degree) or be the results of previous research in mathematics education or "inventions" on the basis of an implicit and holistic pre-understanding of the field of research.

1.2 What Is the Role and Function of Theory in the Research Process?

Theory is not imperialism but simplification. Theory functions in the research process as a means of reducing and controlling the variables that have to be taken into account when the researcher is studying, for example, a didactical process. Mathematics education is a domain in which there are a large number of salient variables. Researchers' patterns of interaction with the reality studied necessarily involve reducing and controlling some of those variables, in order to allow them to attend to those they investigate. Theory is thus one means (the most powerful) we have of making an a priori analysis of the variables inherent in a situation, and hence of allowing us to make rational and defensible choices about what variables to control.

For example, consider the work of the researcher who is creating a classroom situation with the help of didactical engineering. The *a priori* analysis is the study of all possible events (from a cognitive and mathematical point of view) in the development of that situation according to the specific mathematics problems and to the other components of the didactical setting. It corresponds to the "task analysis" of psychologists. This analysis then aids the interpretation of the actual events, so helping our understanding of the process under study.

Theory is also a means of generating questions and problems that are practical, operational, open to empirical study. The same *a priori* analysis of the situation often identifies important phenomena susceptible to investigation. It shapes the view of the reality to be investigated and opens a particular perspective on this reality, hence raising and precising questions and equipping the researcher with a language to formulate these questions. Finally it guides the data collection and the data analysis and is thus a determinant of the research methods to be used.

Sometimes it is appropriate to start with an empirical exploration of the research field without an explicit theoretical model. This happens when the researcher wants to remain open for a deep and adequate understanding of the reality. Reflection on the analysed data can result in a description or an explanatory theory. Theory generated this way is known as "grounded theory", comprising a posteriori accounts emerging from observed data. This notion seems to be consistent with the working definition given above. However it is essential that the process is completed, and that researchers' claims to be developing grounded theory are substantiated in their reports with specific accounts of the emergent theory being provided.

Some more specific functions of theory might actually demand the production of different kinds of theory. Four possibilities are:

- descriptive theories which give an account of what has happened, describe what is the case;
- explanatory theories which seek to explain why or how something happened;
- predictive theories which predict what will happen in given conditions;
- action theories which guide action by identifying what can be done in given conditions.

Identifying such possibilities raises a further question.

1.3 Is It Feasible and Useful to Distinguish Different Kinds of Theory?

Perhaps partial theories might better be designated “models” reserving the word “theory” for the larger frameworks of concepts which offer more global accounts of how things work. It may be noted that the field of mathematics education can be broken down into different kinds of elements and that different elements would demand different global theories. They would also lead to different research questions and different methods of investigation. In the two examples in this section of the proceedings the elements are “didactical situations”(Rouchier) and “meanings of mathematical objects” (Godino).

1.4 How Can Theories in Mathematics Education Be Evaluated?

Mathematics education is not mathematics, nor is it a science. Its theories cannot be proved by an “A implies B” chain of logical reasoning. Nor can we look for the Popperian notion of falsification by a crucial experiment. Rather we must look for self-consistent narratives in terms of identified elements within mathematics education which are sufficient and are illuminating in accounting for the observed phenomena. The criteria by which theories in mathematics education are judged are adequacy and usefulness. Evaluation is therefore essentially pragmatic.

Published theories can also be subjected to external evaluation through data generated independently of the theory. Can possessors of data apply a theory so as to see their observations in a new way? Can generators of theory interpret external data in terms of existing formulations? For example, what light can be thrown on Bagni’s teaching experiment by the theory of didactical situations or by the theory of meanings of mathematical objects? The focus of the paper attends more closely to the elements of the latter theory. A relationship between the introduction of a group concept by means of an historical example and an earlier investigation of the meaning of mathematical objects could be posited. In particular, the introduction of the group structure could be read as “a human activity involving the solution of social-shared problem-situations” leading to the creation of “a symbolic language in which problem-situations and their solutions are expressed”. (Godino & Batanero 1998, p. 179).

By such tests theories are more likely to be modified or extended rather than verified or falsified in their entirety. They may also be extended, modified or changed through theoretical reflection. Experience suggests that major paradigm shifts occur when a theory is found inadequate.

2. Conducting Empirical Research

2.1 What Can Be Learned from Empirical Research?

Empirical research can be carried out in a number of physical and social contexts. In our three examples Bagni reported a teaching experiment carried out in regular classrooms. The textual eigenproductions (TEPs) discussed by Maier were created in classrooms but the data included teachers' responses to these TEPs outside a classroom setting. In Stein's study the social unit was a pair of students working on a task rather than the whole class. The choices implied by these contexts derived from the phenomenon to be researched and the particular questions to be addressed. In all cases the description of the conditions of the study is an essential tool in allowing the reader to interpret the findings.

It was noted that over the history of research in mathematics education, which was seen as a twentieth century phenomenon, there has been a marked shift from quantitative to qualitative methods and approaches. This probably reflects a view that it is the exceptional case which challenges our perceptions of what is the case, and which causes us to seek new interpretations and explanations. We therefore need phenomenological accounts – thick descriptions – of individual cases. On the other hand, a possible danger inherent in the exclusive pursuit of qualitative data is that we may fail to establish the typical to which our special case is the exception. We may need a broader picture, possibly established through quantitative approaches, to anchor our new interpretations.

2.2 What Is the Relationship Between the Researcher and the Researched?

Another issue in empirical research is the effect of the observer on the observed, especially but not exclusively, when the observer is a participant in the situation. In the above discussion of theory it was noted that studying classroom processes becomes possible when we are able to reduce the numbers of variables or to control some of them. It follows that where an external researcher is studying the interactions in a particular classroom the construction of the teaching to be done must be shared by the teacher and the researcher. The technology of building situations (to research) is supported by previous knowledge in the field (including in particular theoretical knowledge). This has been described as ingenierie didactique or didactical engineering.

Action research is research carried out by practitioners on their own practice with a view to changing and improving that practice. Two of the papers (D'Amore & Maier, Stehlíková) described projects which involved teachers taking on aspects of the role of researchers through a sharing of the methods of the projects. Discussion of this led to a further question.

2.3 How Can Research Best Be Shared with Teachers?

There is a responsibility to report on research results in a format which is accessible to teachers. However it is often more fruitful to share the methods of research – usually qualitative methods – with teachers so that they can construct knowledge particular to their own situation.

2.4 Developing Methodologies

What is a methodology?

As with theory there is a question about how global, how all-embracing a methodology must be in order to be called a methodology. In both of the papers above (Marí, Stehlíková) the methodology is detailed and elaborated. In both cases this elaboration performs an integrating function – in the first a particular topic from the school mathematics curriculum is chosen and a very thorough survey of existing research of that topic is synthesised to give a basis for the study. In the second the project examines a range of aspects of learning mathematics and seeks to integrate the work of many researchers including teacher-researchers to produce the findings.

It was agreed that major projects need all-embracing methodologies. However much valuable research is carried on a smaller scale and it was pointed out that without such small scale studies there would be no existing research to survey. What is important is that in reporting research the methods used and the reasons for the choice of methods are reported as clearly as possible to allow others to follow up and build on that work.

3. Conclusions

The conference structure – and themes – worked well for Group 7 so we relate our conclusions to the themes of the conference.

3.1 Communication

All participants found great value in sharing accounts of research practice. We agreed that an important aim (possibly the central aim) of our research is the improvement of mathematics teaching. It follows that an important part of communication is communicating with teachers of mathematics. What we communicate may be research results, but it may be research methods, perhaps helping the reflective practitioner to become a practitioner-researcher.

Whoever the audience of our research reports, in order to maximise communication we need to be as clear as possible about the researchers' underpinning theoretical position, and about both the conditions and the methods of research.

One difficulty with communication with colleagues from different research traditions lies in undeclared assumptions. Ambiguities and misunderstandings may also arise from the different meanings and connotations of words which we use in common.

3.2 Cooperation

The spirit of cooperation was strong in the group.

Like all groups we started from the premise that papers would not be presented, but would have been read by all. This proved to be a valid premise. Nevertheless we found that our A-type sessions allowed a useful focus on individual papers from which authors received feedback and ideas for improving final versions of papers and participants gained deeper insight into what they had read.

3.3 Collaboration

As we worked there was a growing awareness of similarities and differences among our perspectives. It seemed that research paradigms are in some ways particular to the country in which they arise.

Our activities were essentially the first stage of a group project to improve communication by clarifying commonly used terms such as: paradigm, theory, didactics, mathematics.

4. References

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THE ROLE OF THE HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION: REFLECTIONS AND EXAMPLES

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Abstract: *The effectiveness of the use of history of mathematics in mathematics education is worthy of careful research. In this paper the introduction of the group concept to an experimental sample of students aged 16-18 years by an historical example drawn from Bombelli's Algebra (1572) is described. A second sample of students was given a parallel introduction through a Cayley table. Both groups were asked the same test questions and their responses examined and compared.*
Keywords: *history, algebra, group.*

1. Bombelli's *Algebra* (1572) and Imaginary Numbers

Several authors have shown that the history of mathematics can be drawn on by teachers in the presentation of many mathematical topics to the benefit of pupils. It follows that research into the role of history of mathematics in teaching is legitimately considered a part of research into mathematics education (many references can be mentioned; for example: Jahnke, 1991, 1995 and 1996).

Of course we must consider the educational use of history of mathematics at different levels, and these levels can lead to different educational interventions. For example, according to the conception of the mathematics education as *thought transference*, the main purpose of the educational research is *improvement of teaching*. The presentation of mathematical topics using historical references is consistent with this approach. Of course, the effectiveness of the historical introduction will be judged with respect to pupils' learning.

In this paper, we consider an important topic of the curriculum of both high school (for students aged 16-18 years) and undergraduate mathematics, namely the group concept.

Rafael Bombelli of Bologna (1526-1572) was the author of *Algebra*, published twice, in 1572 and in 1579 (the dispute between G. Cardan and N. Fontana Tartaglia about the resolution of cubic equations is well known; let us underline that Scipio Del Ferro was remembered in Bombelli's *Algebra*: manuscript B.1569, Archiginnasio Library, Bologna).

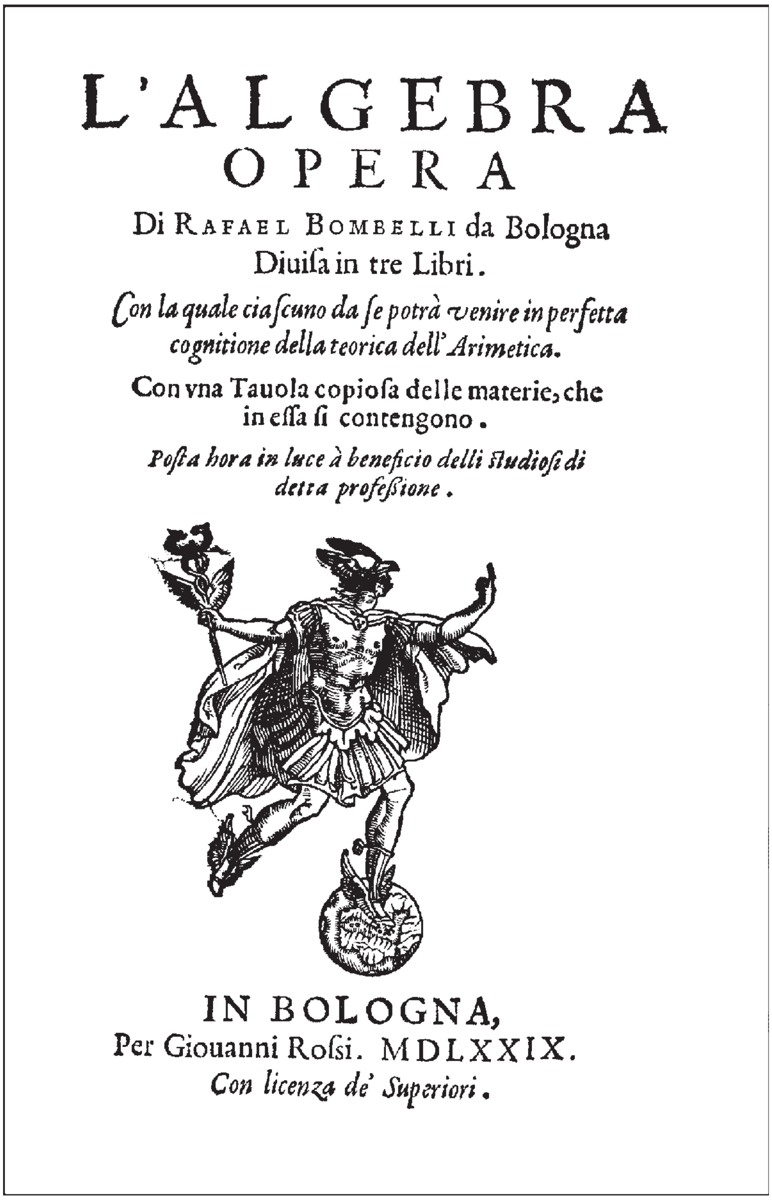
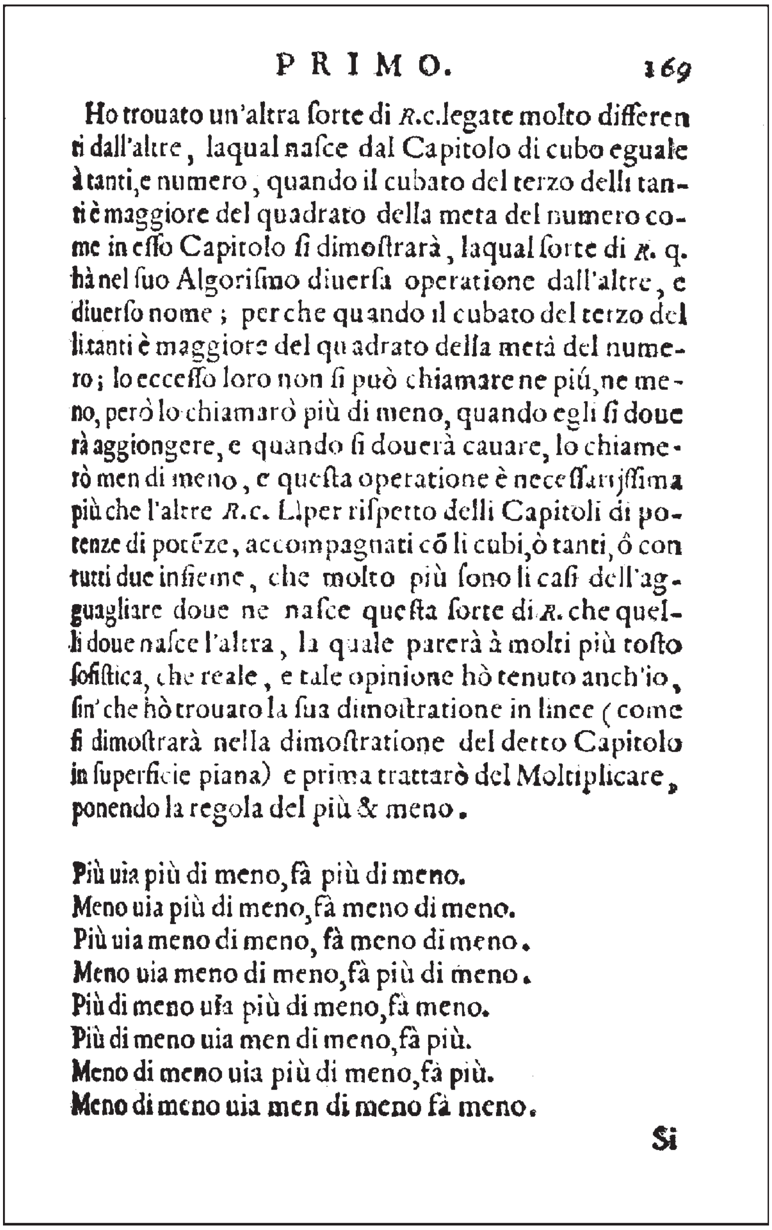

L'ALGEBRA
OPERA
Di RAFAEL BOMBELLI da Bologna
Diuiſa in tre Libri.
Con la quale ciaſcuno da ſe potrà venire in perfetta cognitione della teorica dell'Arimetica.
Con vna Tauola copioſa delle materie, che in eſſa ſi contengono.
Poſta hora in luce à beneficio delli ſtudioſi di detta profeſſione.
IN BOLOGNA,
Per Giouanni Roſsi. MDLXXIX.
Con licenza de' Superiori.

Fig. 1: Bombelli's *Algebra* (1572-1579)

In *Algebra*’s 1st Book, Bombelli introduced the terms *più di meno* (*pdm*) and *meno di meno* (*mdm*) to represent $+i$ and $-i$ and gave some “basic rules”. Let us consider them in Bombelli’s original words (p. 169):

“Più via più di meno, fa più di meno.
Più via meno di meno, fa meno di meno.
Più di meno via più di meno, fa meno
Meno di meno via più di meno, fa più.
Meno via più di meno, fa meno di meno.
Meno via meno di meno, fa più di meno.
Più di meno via men di meno, fa più.
Meno di meno via men di meno, fa meno.”


PRIMO. 169

Ho trouato un’altra ſorte di R.c.legate molto differenti dall’altre, laqual naſce dal Capitolo di cubo eguale à tanti, e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della meta del numero come in eſſo Capitolo ſi dimoſtrarà, laqual ſorte di R. q. hà nel ſuo Algoriſmo diuerſa operatione dall’altre, e diuerſo nome; per che quando il cubato del terzo del li tanti è maggiore del quadrato della metà del numero; lo ecceſſo loro non ſi può chiamare ne più, ne meno, però lo chiamarò più di meno, quando egli ſi douerà aggiongere, e quando ſi douerà cauare, lo chiamerò men di meno, e queſta operatione è neceſſarijſſima più che l’altre R.c. L. per riſpetto delli Capitoli di potenze di poteze, accompagnati cõ li cubi, ò tanti, ò con tutti due inſieme, che molto più ſono li caſi dell’agguagliare doue ne naſce queſta ſorte di R. che quelli doue naſce l’altra, la quale parerà à molti più toſto ſofiſtica, che reale, e tale opinione hò tenuto anch’io, ſin’che hò trouato la ſua dimoſtratione in linee (come ſi dimoſtrarà nella dimoſtratione del detto Capitolo in ſuperficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno, fà più di meno.
Più di meno uia più di meno, fà meno.
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.

Si

Fig. 2: From Bombelli’s *Algebra*

Now, let us translate:

"Più" $\rightarrow$ +1; "Meno" $\rightarrow$ -1; "Più di meno" $\rightarrow$ + i; "Meno di meno" $\rightarrow$ - i;

"Via" $\rightarrow \cdot$ (multiplication); "Fa" $\rightarrow$ = .

So we can write:

$(+1)\cdot(+i)$	=	$+i$	$(-1)\cdot(+i)$	=	$-i$
$(+1)\cdot(-i)$	=	$-i$	$(-1)\cdot(-i)$	=	$+i$
$(+i)\cdot(+i)$	=	-1	$(+i)\cdot(-i)$	=	$+1$
$(-i)\cdot(+i)$	=	$+1$	$(-i)\cdot(-i)$	=	-1

First of all, we underline the importance of the linguistic aspect: for example, we have translated the term "Fa" with the symbol "="; however the modern equality symbol can be referred to a relation "in two directions", while the term "Fa" means that the result of the multiplication in the first member is written in the second member. This can be confirmed by further research.

Moreover, in *Algebra* we find (p. 70):

"Più via più fa più.	Meno via meno fa più.
Più via meno fa meno.	Meno via più fa meno".

We can express these in the following way:

$(+1)\cdot(+1)$	=	$+1$	$(-1)\cdot(+1)$	=	-1
$(+1)\cdot(-1)$	=	-1	$(-1)\cdot(-1)$	=	$+1$

So we can write the following *Cayley table*:

x	**+1**	**-1**	**+ *i***	**- *i***
+1	+1	-1	+ i	- i
-1	-1	+1	- i	+ i
+ *i*	+ i	- i	-1	+1
- *i*	- i	+ i	+1	- i

It can be interpreted as the multiplicative group $(\{+1; -1; +i; -i\}; \cdot)$ of the fourth roots of unity, a finite Abelian group (it is well known that it is a cyclic group and it can be generated either by i either by $-i$; however Bombelli did not notice explicitly this property; moreover he did not notice this for its cyclic subgroup generated by –1).

Of course, in Bombelli's *Algebra* we cannot find a modern introduction either of complex numbers or of the formal notion of a group: Bombelli just indicated some mathematical objects in order to solve cubic equations. These ideas were not immediately accepted following the publication of Cardan's and Bombelli's works. Bombelli himself was initially doubtful and wrote in *Algebra*, p. 169: "I found another kind of cubic root... and I did not consider it real, until I have found its proof"; let us underline that Bombelli considered fundamental the geometric proof of algebraic statements. However the *formal* introduction of i in Bombelli's *Algebra* is important and modern (Bourbaki, 1960, pp. 91-92). We notice, according to A. Sfard, that complex numbers were introduced simply as an *operational* concept:

> "Cardan's prescriptions for solving equations of the third and fourth order, published in 1545, involved [...] even finding roots of what is today called negative numbers. Despite the widespread use of these algorithms, however, mathematicians refused to accept their by-products [...] The symbol [square root of -1 was] initially considered nothing more than an abbreviation for certain 'meaningless' numerical operations. It came to designate a fully fledged mathematical object only after mathematicians got accustomed to these strange but useful kinds of computation" (Sfard, 1991, p. 12).

We can consider the idea of group in a similar way: surely it would be incorrect to ascribe to Bombelli an explicit awareness of the group concept, three centuries before Galois and Dedekind. However, we can state that he (implicitly) introduced – in action – one of the most important concepts of mathematics.

2. Educational Problems: The Focus of Our Work

In an important paper, E. Dubinsky, J. Dautermann, U. Leron and R. Zazkis opened "a discussion concerning the nature of the knowledge about abstract algebra, in particular group theory, and how an individual may develop an understanding of various topics in this domain" (Dubinsky, & al., 1994, p. 267).

Let us now consider only the group concept (in the article quoted we can find interesting considerations of many algebraic notions: the concepts leading up to "quotient group", for instance, are: "group", "subgroup", "coset", "coset product" and "normality": Dubinsky & al., 1994, p. 292). The authors write: "An individual's knowledge of the concept of group should include an understanding of various mathematical properties and constructions independent of particular examples, indeed including groups consisting of undefined elements and a binary operation satisfying the axioms" (Dubinsky & Al., 1994, p. 268; Leron & Dubinsky, 1995; as regards the teaching procedure using the computer software ISETL, presented in Dubinsky & Al., 1994, see for example: Dubinsky & Leron, 1994).

In a recent paper (1996), B. Burn strongly emphasises that the notion of group in Dubinsky & al. (1994) is introduced by formal definitions. In particular, a group is "a set with a binary operation satisfying four axioms [...] They espouse a set-theoretic viewpoint" (Burn, 1996, p. 375). Then Burn notes that "a set-theoretic analysis is a twentieth-century analysis performed upon the mathematics of earlier centuries as well as our own" (Burn, 1996, p. 375); so he suggests a pre-axiomatic start to group theory (Burn, 1996, p. 375; for example, the author quotes: Jordan & Jordan, 1994). So, according to Burn, the group concept can be introduced *before* offering axioms (he points out the importance of geometric symmetries: Burn, 1996, p. 377; Burn, 1985; as regards fundamental concepts of group theory he quotes: Freudenthal, 1973).

In another recent paper (1997), E. Dubinsky & al. assert that "seeing the general in particular is one of the most mysterious and difficult learning tasks students have to perform" (Dubinsky & al., 1997, p. 252; see also Mason & Pimm, 1984); the authors mention students' difficulties with permutations and symmetries (Asiala & Al., 1996; they quote moreover: Breidenbach & Al., 1991; Zazkis & Dubinsky, 1996; as regards visualisation, see: Zazkis & Al., 1996). So a very important question is the following: is it possible (and useful) to introduce the group concept by a pre-axiomatic first treatment?

In this paper we do not claim to give a full answer to this question: rather we attempt to contribute to knowledge of the development of students' understanding of the group concept, with reference to historical examples. For instance, can the consideration of the group mentioned above help students in the comprehension of the group concept? In particular, will consideration of Bombelli's "basic rules" bring *all* the properties that are fundamental to the group concept to pupils' awareness?

As indicated above, we operate on teaching to improve its quality by *thought* transference; but some reactions, especially those in pupils' minds, are inferred, they are plausible rather than certain. We proposed an example in the historical sphere, in order that students will "learn" in this sphere, but so that the knowledge achieved will *not* be confined to the historical sphere: evolution to different spheres is necessary. A problem that can limit the efficacy of mathematics education as (merely) *thought transference* is as follows: if we operate (only) on teaching, are we *sure* that a correct evolution will take place in the students?

In what follows we shall examine students' behaviour in response to a comparative teaching experiment carried out with two samples of high school students. To the first sample, we simply quoted Bombelli's "basic rules"; in the second one we gave the *Cayley table*. We wanted to find out if the four properties used in the definition of group (according to Burn, they are introduced by the terms: "closed", "associative", "identity" and "inverse": Burn, 1996, p. 372) are acquired by students from these introductions.

3. The Group Concept from History to Mathematics Education

The sample comprised the students of three classes from the 3rd *Liceo Scientifico* (pupils aged 16-17 years), 68 pupils, and of three classes from the 4th *Liceo scientifico* (pupils aged 17-18 years), 71 pupils (total: 139 pupils), in Treviso (Italy). At the time of the experiment, pupils knew the definition of i ($i^2 = -1$); they did *not* know the group concept.

We divided (at random) every class into two parts, referred to as A and B; then we gave the following cards to the students. In the card given to the students in part A (total 68 students) we just quoted Bombelli's "basic rules"; in the card given to the students of the part B (total 71 students), we did not quote Bombelli's "basic rules" but we gave *Cayley table*. (See Cards A and B illustrated below.) By the test questions, we wanted to find out if consideration of a simple historical example (without an axiomatic set-theoretic viewpoint) is useful in introducing the group concept; moreover, we wanted to find out the difference between the results obtained by students who were given a group description by Bombelli's "basic rules" and the results obtained by students who had received the *Cayley table*.

As regards the four properties used in the definition of a group, we expected that closure, associativity and the presence of the unit in the set G = {+1; –1; +*i*; –*i*} would be apparent to many pupils. Closure appears clearly both in Bombelli's rules and in the *Cayley table*; the other properties are quite familiar. The inverse property namely that, for every $x \in G$, there is an element $x' \in G$ such that $x \cdot x' = x' \cdot x = 1$ can be harder to discern.

Card A. In *Algebra*, Rafael Bombelli of Bologna (1526-1572) gave the rules:

$(+1)\cdot(+1) = +1$	$(-1)\cdot(+1) = -1$	$(+1)\cdot(-1) = -1$	$(-1)\cdot(-1) = +1$
$(+1)\cdot(+i) = +i$	$(-1)\cdot(+i) = -i$	$(+1)\cdot(-i) = -i$	$(-1)\cdot(-i) = +i$
$(+i)\cdot(+i) = -1$	$(+i)\cdot(-i) = +1$	$(-i)\cdot(+i) = +1$	$(-i)\cdot(-i) = -1$

(we have written the rules by modern symbols).

Consider the set G = {+1; –1; +*i*; –*i*}. Are the following statements true or false?

(1) The product of two elements of G is always an element of G.

(2) The multiplication of elements of G is associative.

(3) There is an element $e \in G$ such that, for every $x \in G$, $e \cdot x = x \cdot e = x$.

(4) For every $x \in G$, there is an element $x' \in G$ such that $x \cdot x' = x' \cdot x = e$.

Card B. Let us consider the following table:

x	+1	-1	+ *i*	- *i*
+1	+1	-1	+ *i*	- *i*
-1	-1	+1	- *i*	+ *i*
+ *i*	+ *i*	- *i*	-1	+1
- *i*	- *i*	+ *i*	+1	- *i*

Consider the set G = {+1; –1; +*i*; –*i*}. Are the following statements true or false?

(1) The product of two elements of G is always an element of G.

(2) The multiplication of elements of G is associative.

(3) There is an element $e \in G$ such that, for every $x \in G$, $e \cdot x = x \cdot e = x$.

(4) For every $x \in G$, there is an element $x' \in G$ such that $x \cdot x' = x' \cdot x = e$.

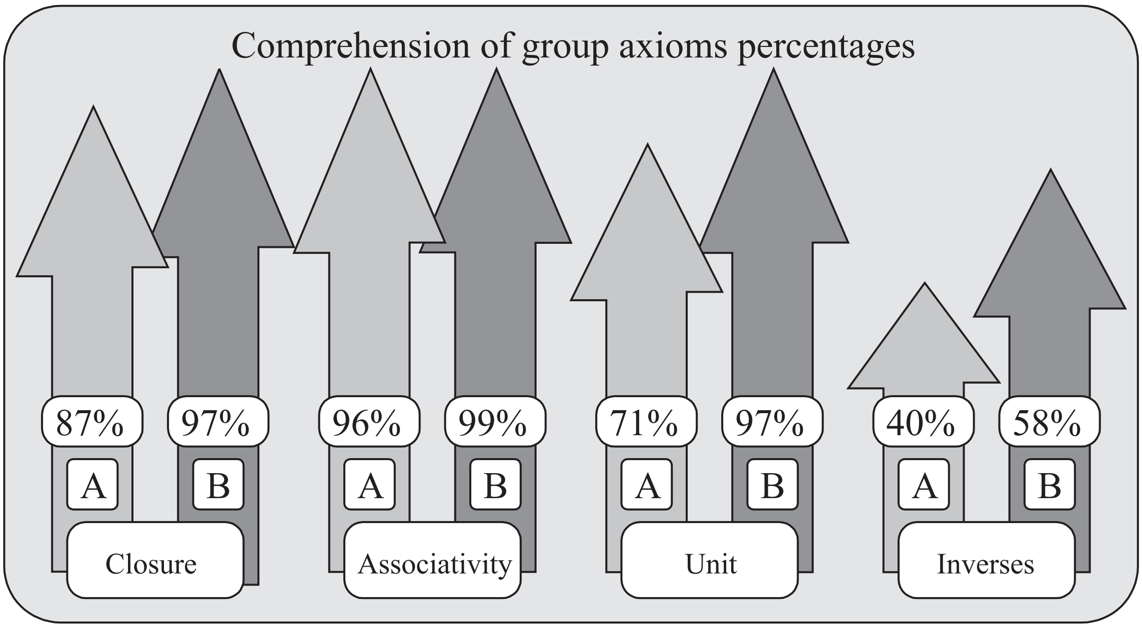
The time allowed to read the card and answer the questions was 10 minutes. (We wanted students to examine the problem ‘at a glance’). The students answers were as follows for the two samples.

Card A	true		false		no answers	
(1)	59	87%	2	3%	7	10%
(2)	65	96%	1	1%	2	3%
(3)	48	71%	3	4%	17	25%
(4)	27	40%	21	31%	20	29%

Card B	true		false		no answers	
(1)	69	97%	2	3%	0	0%
(2)	70	99%	0	0%	1	1%
(3)	69	97%	0	0%	2	3%
(4)	41	58%	8	11%	22	31%

These results indicate that properties 1, 2, 3 (closure, associativity, presence of the unit element) are recognised by students. As regards properties 1 and 2 the differences between sample A and sample B are fairly small, with a greater difference for property 3: it seems that the *Cayley table* was somewhat more helpful to pupils than Bombelli’s rules. As regards property 4 (the presence of inverses) the situation is rather different: only 40% of the students (sample A) and 58% (sample B) accepted the property. Let us represent some of the results in Tab. I (it is just a qualitative representation: the differences between percentages in test A and in test B are sometimes slight).

We interviewed individually those pupils (A: 41 students; B: 30 students) who did not consider statement 4 to be “true”. Almost all these students simply stated that they did not realize the presence of the inverses just by examining Bombelli’s rules (or the *Cayley table*). So consideration of the historical example was successful in causing all the conjectured reactions and expected effects for only some of the students (let us notice that it is well known that a multiplicative *finite* submonoid G of the multiplicative group **C*** of non-zero complex numbers is a subgroup: the sufficiency of the closure test, in this case, is underlined also in: Burn, 1996, p. 373; so, as regards the previous tests, it is inconsistent to state that properties 1, 2, 3 are true and property 4 is *not* true; however, high school pupils cannot know the mentioned proposition: Dubinsky & Al., 1997, p. 251).



Tab. I

4. History of Mathematics and Epistemology of Learning

The consideration of relevant examples from the history of mathematics can really help the introduction of important topics. As regards the group concept, however, the supposed reactions took place completely only for *some* students.

A pre-axiomatic start to group theory can be useful (see for example: Jordan & Jordan, 1994), but it is not always enough to assure full learning. As indicated above, the question remains open. It is possible to object that the mere offering of Bombelli's rules is insufficient to achieve a complete learning of the group concept. Let us emphasise that our research was an exploratory study. For a fuller investigation it would be necessary to identify clearly sampling criteria and pre-course intuitions (as underlined in: Burn, 1996, p. 371).

Let us quote once again Dubinsky & al. (1997):

> "A historical view is useful in designing research and instruction with respect to group theory. History is certainly a part of our methodology, but we are influenced not only by the record of who proved what and when, but also with the mechanisms by which mathematical progress was made"
> (Dubinsky & al., 1997, p. 252; according to Piaget & Garcia, 1983, there is a close connection between historical and individual development at the level of cognitive mechanism; see: Dubinsky & Al., 1997, p. 252).

The main limitation of the notion of mathematics education as *thought transference* lies in the uncertainty about real effects (upon the learning) of teachers' choices. We make no claims for the teaching of abstract algebra – through the consideration of historical references or otherwise – as regards the nature and the meaning of mathematical objects. Here several problems are opened (Godino & Batanero, 1998), involving several fundamental philosophical questions. However it is important and necessary to control the educational research process by experimental verification: this can profoundly affect the delineation of the research and give it an important, particular epistemological status.

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THE MEANINGS OF MATHEMATICAL OBJECTS AS ANALYSIS UNITS FOR DIDACTIC OF MATHEMATICS

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Abstract: *In this report we argue that the notion of meaning, adapted to the specific nature of mathematics communication, may serve to identify analysis units for mathematical teaching and learning processes. We present a theory of meaning for mathematical objects, based on the notion of semiotic function, where we distinguish several kinds of meanings: notational, extensional, intensional, elementary, systemic, personal and institutional. Finally, we exemplify the theoretical model by analysing some semiotic processes which take place in the study of numbers.*
Keywords: *ontology, semiotics, mathematics education.*

1. Introduction

According to Vygotski (1934), the unit for analysing psychic activity - which reflects the union of thought and language - is the meaning of the word. This meaning - conceived as the generalisation or concept to which that word refers - englobes the properties of the whole, for which its study is considered, in its simpler and primary form. We think that the search of analysis units for mathematical teaching and learning processes should also be focussed on the meaning of the objects involved.

However, the notion of meaning should be interpreted and adapted to the nature of mathematical knowledge and to the cognitive and sociocultural processes involved in its genesis, development and communication. We agree with Rotman (1988) in that it is possible and desirable to develop a specific semiotic of mathematics, which would take into account the dialectics between mathematical sign systems, mathematical ideas and the phenomena, for the understanding of which they are built, within didactic systems.

To reach this aim, we consider it necessary to elaborate a notion of meaning specifically adapted to didactics, by interpreting and adapting existing semiotical and epistemological theories. This need was forecast by Brousseau (1986 p. 39), who wondered whether there is or there should be a notion of meaning, unknown in linguistics, psychology, and mathematics, though especially appropriate for didactics.

In this research work, we present some elements of a semiotic model, specific to Didactic of Mathematics, starting out from the notion of semiotic function proposed by Eco (1979), and classifying mathematics entities into three types: extensional, notational and intensional ones. Based on the different nature of these types of mathematical entities and the contextual factors conditioning mathematical activity, we identify meaning categories to describe and explain the interpretation and communication processes taking place in the heart of didactic systems.

The notion of meaning -conceived as the content of semiotic functions and applicable to mathematical terms and expressions, as well as to conceptual objects and problem situations - allows us to identify analysis units for mathematics teaching and learning processes. We finally describe some of these units, applying them to analyse examples of semiotic processes at the different stages of the study of whole numbers.

2. Meaning as the Content of Semiotic Functions

We use the term ‘meaning’ according to the theory of semiotic functions described by Eco (1979), and consider the pragmatic context as a conditioning factor of such semiotic functions. Here, this context includes the set of factors sustaining and determining mathematical activity, and therefore, the form, appropriateness and meaning of the objects involved. According to Eco, “*there is a semiotic function when an expression and a content are in correlation*” (Eco 1979 p.83). Such a correlation is conventionally established, though this does not imply arbitrariness, but it is coextensive to a cultural link. There may be functives of any nature and size. The original object in the correspondence is the signifiant (plane of expression), the image object is the meaning (plane of content), that is, what it is represented, what it is meant, and what is referred to by a speaker.

In a previous article (Godino and Batanero 1998) we analysed the emergence of mathematical objects from the meaningful practices carried out by persons or institutions when solving specific problem fields. A meaningful practice is defined as "a manifestation (linguistic or not) carried out by somebody to solve mathematical problems, to communicate the solution to other people, to validate and generalise that solution to other contexts and problems ", and the meaning of a mathematical object is identified as the system of practices linked to the field of problems from which the object emerges at a given time.

Since semiotic functions are established by a person in a given context with a communicative or operative intention, for us these functions can be considered as meaningful practices and reciprocally, behind each meaningful practice we could identify a semiotic function or a lattice of semiotic functions.

Meaningful practices, conceived as intentional actions mediated by signs, might be the basic units for analysing cognitive processes in mathematics education.

3. Notational, Extensional and Intensional Meanings

We can establish semiotic functions between three primitive types of mathematics entities:

- Extensional entities are the problems, phenomena, applications, tasks, i.e., the situations which induce mathematical activities.
- Notational entities, that is, all types of ostensive representations used in mathematical activities (terms, expressions, symbols, graphs, tables, etc.)
- Intensional entities: mathematical ideas, generalisations, abstractions (concepts, propositions, procedures, theories).

In Godino and Recio (1998), we analyse to some extent the nature of these entities interpreting and adopting ideas by Freudenthal, Vergnaud and Dörfler. Notational entities play the role of ostensive and essential support that makes mathematical work possible, because generalisations and situation problems are given by notational systems, which describe their characteristic properties. Abstractions are not directly observable and problem situations are frequently used to provide abstract mathematical objects with a context. Abstractions and situations are neither inseparable from the notations (ostensive objects) which embody them, nor identifiable with them, that is, we consider that mathematics cannot be simplified to the language which expresses it.

The three types of primary entities considered (extensional, intensional and notational) could perform both roles of expression or content in semiotic functions. There are, therefore, nine different types of such functions, some of which may clearly be interpreted as specific cognitive processes (generalisation, symbolisation, etc.).

In this paper, we classify and characterise these functions as regards to the content (meaning) involved, so that the nine types are reduced to the following three:

(1) *Notational meaning*: Let us call a semiotic function notational when the final object (its content), is a notation, that is, an ostensive instrument. This type of function is the characteristic use of signs to name world objects and states, to indicate real things, to say that there is something and that this is built in a given manner. The following examples demonstrate this type of meaning:

- When a particular collection of five things are represented by the numeral 5.
- The symbol P_n (or n!) represents the product n(n-1)(n-2)...1

(2) *Extensional meaning*: A semiotic function is extensional when the final object is a situation problem, as in the following examples:

- The simulation of phenomena (i.e., it is possible to represent a variety of probabilistic problems with urn models).

(3) *Intensional meaning*: A semiotic function is intensional when its content is a generalisation, as in the following examples:

- In expressions such as, “Let μ be the mathematical expectation of a random variable ”, or “ Let f (x) be a continuous function”. The notations μ, f(x), or the expressions ‘mathematical expectation’, ‘random variable ‘ and ‘continuous function’, refer to mathematics generalisations.

Furthermore, all intensional and extensional functions imply an associated notational function, since abstractions as well as problem situations are textually fixed.

4. Elementary and Systemic Meanings

An elementary meaning is produced when a semiotic act (interpretation / understanding) relates an expression to a specific content within some specific space-temporal circumstances: It is the content that the emitter of an expression refers to, or the content that the receiver interprets. In other words, what one means, or what the other understands. Examples of this use of the word 'meaning' are the deictic signs, rigid designations (Eco 1990) where the content is indicated by gestures, indications or proper names. The content of the semiotic function is a precise object, which may be determined without ambiguity in the spatial-temporal circumstances fixed.

The semiotic processes involved in building mathematical concepts, establishing and validating mathematics propositions, and, as a rule, in problem solving processes, yield systemic meanings. In this case, the semiotic function establishes the correspondence between a mathematical object and the system of practices which originates such an object (Godino and Batanero 1998). The structural elements of this systemic meaning would be the problem situations (extensional elements), the definitions and statements of characteristic properties (intensional elements) and the notations or mathematical registers (notational elements). These three types of primitive entities provide a classification of practices constituting mathematical abstractions.

5. Personal and Institutional Meanings

The theoretical nature of systemic meanings and encyclopaedias tries to explain the complexity of semiotic acts and processes, but they are not fully describable. Practice systems differ substantially according to the institutional and personal contexts where problems are solved. These contexts determine the types of cultural instruments available and the interpretations shared, and therefore the types of practices involved.

> "*Even when, from a general semiotic viewpoint, the encyclopaedia could be conceived as global competence, from a sociosemiotic view is interesting to determine the various degrees of possession of the encyclopaedia, or rather the*

partial encyclopaedias (within a group, sect, class, ethnic groups, etc" (Eco 1990, p. 134).

Due to these characteristics of the systemic meanings, we consider it necessary to distinguish between *institutional meanings* and *personal meanings*, depending on whether practices are socially shared, or just idiosyncratic actions or manifestations of an individual. In the second case, when the subject tries to solve certain classes of problems, he builds a personal meaning of mathematical objects. When this subject enters into a given institution (for example, the school) he/she might acquire practices very different from those admitted for some objects within the institution.

A matching process between personal and institutional meanings is gradually produced. The subject has to appropriate the practice systems shared in the institution. But the institution should also adapt itself to the cognitive possibilities and interests of its potential members.

The types of institutions interested by a specific class of mathematical problems might be conceived as communities of interpreters sharing some specific cultural instruments and constitute a first factor for conditioning the systemic meanings of mathematical objects.

6. Semiotic Acts and Processes in the Study of Numbers

In this section, and to give examples of the theoretical concepts described, we apply the semiotic model outlined to the analysis of some semiotic acts and processes involved in the study of whole numbers.

6.1 Elementary Meanings: The First Encounter With Numbers

The first encounter with numbers for most children, is produced at pre-school age, when their parents teach them the series of words 'one', 'two', 'three', etc. to count small collections of objects: hand fingers, balls, sweets, etc. Afterwards, they will find school tasks close to those reproduced in Fig.1, which have been taken from a book for1st year primary teaching.

The text is intended to make the child recognise and write the numerals ‘1’, ‘2’, ‘3’, ..., at the same time as different collections of objects represented are assigned to the corresponding numerical symbol. From the drawing of a head, a sun, a cat an arrow points at to the symbol 1. As an exercise, drawing 1 beside a flower and a moon is implicitly requested. A similar method is used for teaching the number 2, its form, writing, and use.

In the tasks proposed, we can identify the three classes of objects and semiotic functions which characterise mathematical activity, according to our semiotic-anthropological model: Notations (ostensive instruments), extensions, and generalisations (or abstractions).

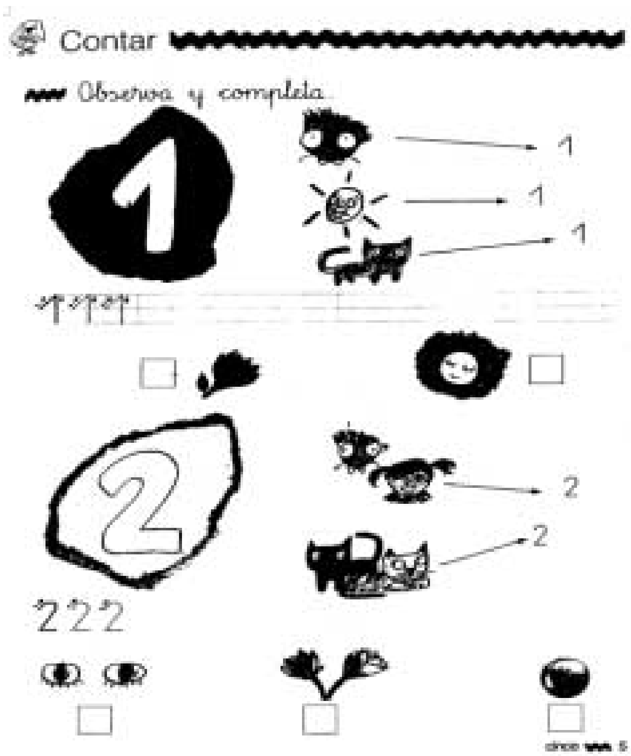

Fig. 1: Learning the numbers 1 and 2

In fact, the concrete object drawings (head, sun, cat, flowers, eyes, etc.) are iconic representations of such objects; the meaning of the icons is the corresponding concrete object (extensional meaning); the schemes implicitly suggest answering the question, “How many objects are there?”, so that the reference context is not just made up by mere concrete objects, but rather the problem situation of computing the cardinal. It is important to observe that, in the task proposed by the book, the physical objects are not in fact present. Therefore, the immediate reference context to which numerical symbols refer is a world of ostensive representation (textual). This fact implies additional semiotic complexity, and hence, interpretative effort by the child. In this series of tasks, we also identify two operative invariants (generalisations):

- the same symbol,’1’, and the number-word ‘one’ are associated to various drawings of unitary collections;
- the symbol, ‘2’, and the number-word ‘two’ are associated to different pairs of objects. The mental objects (or better, the logical entities), one and two, are implicitly evoked as from the first teaching levels. The tasks aim, in psychological terms, is the progressive construction of the objects, number one, two, etc. in the child’s mind. In anthropological terms it is the child’s acquisition of the habit of naming any collections by using the series of number-words, and the series of numerical-symbols (numerals).

We also identify the following semiotic functions (acts and interpretation processes):

I1: The drawing of concrete objects is implicitly related to the concrete objects (extensional meaning) they represent.

I2: The object collections are interrelated with the numerical symbols, ‘1’, ‘2’ (notational meaning).

I3: The number-words ‘one’, ‘two’, are associated (implicitly) with the numerical symbols, ‘1’, ‘2’ (notational type).

I4: Each icon collection, and its associated numerical symbol, is implicitly interrelated with the corresponding mathematical concepts (one, two, three, ...) (intensional semiotic function).

The interpretation of these semiotic functions requires specific codes and conventions which should be known and interpreted by the receiver of the message (the child), to successfully complete the tasks and to progressively acquire the notion of number. We point out that the spaces left between the different icons play a fundamental role, since they inform which objects should be counted in each case.

The writing guides for learning to draw the numerals ‘1’ and ‘2’ are also full of symbolisms, which they are difficult to decode without the teacher’s assistance. They graphically present rules such as: “*Do this drawing in the way I show you and repeat it several times*”.

This type of rules poses the subject with a problem situation (or simply, a routine task), and involves an extensional meaning, according to our theoretical model.

Our analysis shows the multiplicity of codes for whose recognition the children will require a systematical teacher’s assistance. This supports Solomon’ s thesis (1989 p. 160), that “*knowing number should be reconceptualized as involving entering into the social practices of number use*”, and not as an issue of individual construction of the necessary and sufficient logical structures for understanding numerical concepts.

6.2 Systemic Meanings of Numbers

In the previous section we have shown examples of notational, extensional and intensional elementary meanings involved in the study of numbers at elementary school. The organised set of these elementary meanings would correspond to what we call a number systemic meaning, which would be personal or institutional depending on whether we take a particular subject (a child) or an institution (community of interpretants) as a reference.

The systemic meaning of numbers within a given teaching level (school institution) is determined by curricular documents, school textbooks, and by the teachers’ own preparation of their lessons on a mathematical topic.

We can observe that the meaning of numbers in curricular documents, is described in an encyclopaedic or systemic form, since it refers to a complex of situational, intensional and notational elements. Numerical competence will be achieved through carrying out an organised practices system of progressive complexity throughout a prolonged period of time, which is extended beyond primary teaching.

The institutional systemic meaning of numbers described in the curricular documents, will later be interpreted by the textbook authors and by the teachers themselves when designing their didactic interventions, to select and fulfil the practices they consider most appropriate to their institutional circumstances. These practices will finally be carried out by the pupils, and will determine the personal meanings that these pupils progressively build.

At primary school, numbers are some "special symbols", 1, 2, 3, ... associated to collections of objects, to count, order, and name them. Children may also carry out activities with concrete materials (rods, toothpicks, multibase blocks, abaci) which constitute more primitive numbering systems than the place-value decimal numbering system, privileged by mathematical culture due to its efficiency. It is not rare, therefore, that if we ask a child, "*What are numbers?*", he/she will answer, at best:
"*They are symbols, 0, 1, 2, 3,. ...., invented by man to count and compare quantities. These symbols form the set of numbers*".

At elementary school, numbers are neither 'the cardinal of finite sets', nor 'the common property to all finite sets mutually coordinable. Few people (children, adults, even teachers) would provide such a description of numbers; however, they handle numbers, know to use them effectively for counting and ordering. For these people, numbers have a different meaning from that shared by professional mathematicians.

Even for professional mathematicians the descriptions of numbers may vary substantially. In Cantorian mathematics, whole numbers "are the elements of the quotient set determined on the set of finite sets by the relationship of equivalence of coordinability between sets". However, for Peano's mathematics, a totally ordering set will be called whole numbers if it fulfils the following conditions:

- Any successor of an element of N belongs to N.
- Two different elements of N cannot have the same successor.
- There is an element (0) that it is not a successor of any other element in N.

- All subsets of N that contain 0 and contain the successor of each one of their elements coincide with N.

Therefore, there is no single definition of whole number set, but rather various, adapted to problem situations, intentions and semiotic tools available in each particular circumstance. Each definition we may make for whole numbers emerges from a specific practices system, hence, it involves a class of problem situations and specific notational systems. In principle, each phenomenological numerical context (sequence, counting, cardinal, ordinal, measure, label, number writing, computation) can produce an idiosyncratic meaning (or sense) for numbers. Institutional contexts also share idiosyncratic practice systems and, consequently, they determine differentiated meanings.

7. Conclusions and Implications

The idea guiding our work is the conviction that the notion of meaning, in spite of its extraordinary complexity, may still play an essential role to as a basis for research into the didactic of mathematics. We think that an anthropological approach to this discipline, as Chevallard (1992) proposes, complemented with specific attention to semiotic processes, may help us to overcome a certain transparency illusion about mathematics teaching and learning processes, showing us the multiplicity of codes involved and the diversity of contextual conditioning factors. The construct of *systemic meaning* postulates the complexity of mathematical knowledge by recognising its diachronic and evolutionary nature. This makes us aware of the relevance of semiotic-anthropological analysis of problem fields associated to each knowledge, their structure variables, and the notational systems used, since knowledge emerges from people's actions when faced with problem situations, as mediated by the semiotic tools available.

Hence, it may be useful in curricular design, development and evaluation as a macro-didactic unit of analysis, guiding the search and selection of representative samples of practices characterising mathematical competence.

On the other hand, and taking into account the complex nature of the meaning of mathematical objects, we should often focus attention on specific interpretative processes and on the inherent difficulties of the same when analysing student and teacher classroom performances. The construct of *elementary meaning* and the description of its various types permits us to focus attention on the implicit codes which condition acts and processes of understanding in mathematics education. This will be useful for identifying critical points, conditioning factors of semiotic acts and processes in mathematical activity and anticipating didactical actions.

The meaning of a mathematical object has a theoretical nature and cannot be totally and unitarily described. Practices carried out to solve mathematical problems differ substantially according to institutional and personal contexts. For this reason, we introduce institutional meanings to distinguish between these different points of view and uses on the same mathematical object. These practices are interpreted in this article as semiotic functions or sequences of semiotic functions, analysing their types, and taking into account the nature of their content (extensional, intensional and notational).

Personal meanings are built by the individual subject - what he/she learns, his/her personal relation to the object - and do not just depend on cognitive factors, but rather on the semiotic-anthropological complex in which this relation is developed, that is, on the element of meaning and the dialectic relationships between them as they are presented.

The theoretical model outlined intends to facilitate the study of the relationship between personal and institutional meanings of mathematical objects. It also implies a strong support for conceiving mathematics and its teaching and learning as a social practice. Children's learning difficulties, errors or failures, are explained by their different interpretations of each situation with respect to what the teacher intended, or simply by their lack of familiarity with the situation. "Not having entered into the social practices of a particular situation, subjects are lost about how to act, though they make the best of it". (Solomon 1989, p. 162).

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Acknowledgment

This research has been supported by the DGES (MEC, Madrid), Project PB96-1411.

DIDACTICAL ANALYSIS: A NON EMPIRICAL QUALITATIVE METHOD FOR RESEARCH IN MATHEMATICS EDUCATION

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Abstract: *The methods used for the research in Mathematical Sciences Education are common for Psychology, Pedagogy and other related fields. But, to be honest, many of these approaches lead to too punctual results, not very important and with few possibilities to modify substantially the educational practice. The reason could lies in the inadequacy of such methods to cover the complexity of the field, in which many factors are operating in a non isolated way and interconnected to each other through relationships which need to be identified and analysed previously in a global framework under the mathematical knowledge as a common factor. To carry out this previous task, about which we must decide later the usage of the most suitable methods, we have been using a non empirical integrating research procedure, called Didactical Analysis, built up in the conjunction of meta-analysis and qualitative approaches. This paper exposes the principles, the conceptual framework and the techniques that make up this still being studied methodology.*
Keywords: *methodology, meta-analysis, epistemology of mathematics.*

1. Introduction

The methodology deals, in a general way, with *the way we get knowledge about the world* (Denzin; Lincoln, 1994). In this case, it deals with the world of the phenomena in the educational processes of teaching and learning of Mathematics. The study of these phenomena can be raised by following a general process which Romberg (1992, p. 51) summarizes in ten steps represented in bold line's squares in figure 1. From these ten main activities, the author attaches special importance to the first four ones, while the last six ones (from 5 to 10) have to do with the operating or technical part of the process. But, as long as there are more than twenty recognized procedures for developing this second part, we find that the activities expected to be the most important ones in the process seem to be supported, solely and without any added specifications, by the intuition and researcher's skills, by the background of the problem to be researched (Bishop, 1992, pp. 712-714) and by the previous knowledge coming from the scholarly community. These considerations would be enough to guarantee the quality and relevance of the results if the phenomena were not so complex as they really are, if the main factors of that complexity took part of the research in terms of relationships and, finally, if scholars bore in mind a kind of integration of the different perspectives and traditions beyond the mere interdisciplinary approach. But these conditions either are not carried out satisfactorily or there are doubts that it would be like that, which may lead to the so called "feet of clay" research, that is to say, faultless from the operative or technical point of view (activities 5 to 10) but faulty regarding their initial fundamentals and assumptions (activities 1 to 4).

To try to improve this situation in those researches involving a specific mathematical knowledge, we propose to introduce a mechanism of systematic control of the research process fundamentals just between activities 3 and 4 of Romberg´s scheme. It consists in three activities (3.1, 3.2 and 3.3, figure 1) involving a method that we have recently built and used with acceptable results (González 1995, 1998; Ortiz 1997) looking for an adequate response to the complexity and specificity of Educational phenomena in Mathematics.

The procedure is based on the general principles of *meta-analysis* (Glass & McGaw & Smith 1981), the *multivocal revision* (Ogawa & Malen 1991) and what is known as *conceptual analysis* (Scriven 1988). For its name we have chosen the term "didactical analysis", used in other sense to describe "*... the analysis of the mathematics contents, which is carried out at the service of the organisation of its teaching in the educational systems...*"(Puig, 1997, p. 61). In this case we refer to a systematic, genuine and integrating process which gives an own personality to the research in its first steps.

The modification proposed is justified by the following arguments, which will be extensively developed in the next sections: a) the complexity of phenomena, in which many factors are interacting and for which analysis many perspectives and procedures are needed; b) the partial specificity of the field of study, founded in the involvement of mathematical knowledge; c) the lack of the interdisciplinary approach and the need of an integration of knowledge and approaches in order to achieve a greater effectiveness.

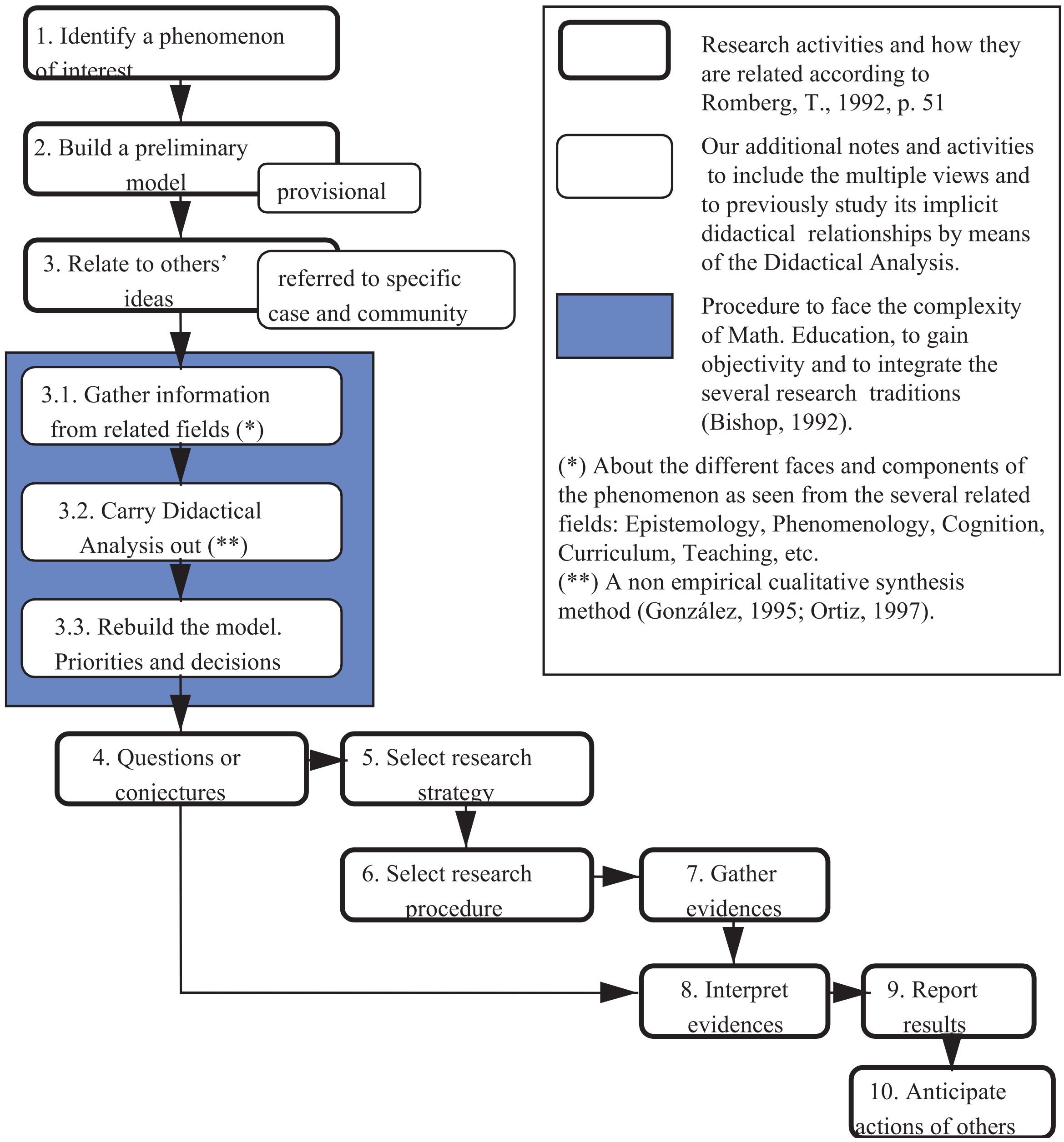

Fig. 1: Research activities and how they would be related according to our experience

2. Mathematics Education: A Field of Relationships vs. Related Fields

In Mathematical Sciences Education, a series of areas, which in the educational practice interact and operate together, can be identified and theoretically separated. Of all these areas we can emphasize, firstly, the one that has to do with *cognitive aspects*, covering, among others, the characteristics and evolution of learns, errors, difficulties and representations and the acquisition of automatism and skills.

On the other hand, we can find a field focussed on teaching, in which specific aspects such as the following: nature, relationships, structure and organisation of the school curriculum (objectives and other organizers (Rico, 1997, pp. 39-59)) witnessing complex factors and conditions (sociocultural, economical, etc.), educational policies, curricular projects and teachers training, can be identified.

Thirdly, we can distinguish a parcel, which is more connected to the real teaching and learning processes, in which some interactions among different factors from the two previous fields are taking part at different levels (design, planning and implementation) (Coriat 1997, pp. 156-157), that is to say: methods for improving learning; resources and materials and curricular adaptations, for instance.

The separation between the above mentioned three fields, which can mainly be observed in the preponderance that researches give to the psychological or pedagogical aspects, seems to be an inadequate approach. The mentioned fields are interconnected to each other and specially related to the Psychology of Mathematics Education (Fischbein, 1990, pp. 6-12) in a first level focussed on the educational finalities and on the general characteristics of mathematical knowledge. Secondly, in special when a specific mathematical topic is taking part, that first level has a tight dependency from other basic elements, as it is the case of Mathematics, its Epistemology and its History or of the Phenomenology of mathematical knowledge; a second level of dependency, which is focussed on both finalities and mathematical contents and based on the following general principles (made up from the considerations of Tymoczko (1986), Davis & Hersh (1988) and Puig (1997)):

a) Mathematical knowledge is perfectible, partial and incomplete, being exposed to failures and having to do with ideas or conceptual objects to which humans accede through the discovery or the non-arbitrary invention or creation. These objects are independent of its symbolisation, having a conventional existence and sharing two different areas: the individual conceptual and the supra individual, cultural or collective as a part of the shared conscience.

b) The phenomena that organize the mathematical concepts are the objects, its properties, the actions on them and the properties of such actions, belonging, all of them, to an unique world in expansion containing the results from human cognition and, in particular, the results from the mathematical activity (Puig, 1997, p. 67).

c) The creation / discovery of mathematical knowledge is conditioned by the common factors to all individuals and cultures that make it possible: the common characteristics of human mind (physiological, for instance), of the environment (physical, social, cultural, for instance) and of the interaction between them.

From the principles and its relationships we draw the following consequences:

1. The involvement of the three factors (mind, environment and interaction) took part in all the interpretations about nature and the way of production of mathematical knowledge, so that the epistemological analysis must take its characteristics into account.

2. The analysis of mathematical knowledge from an educational perspective must include the epistemological, cognitive and phenomenological analysis, which would be related to the sociocultural aspects as well as to teaching and curricular issues as specific and terminal subjects in Mathematical Sciences Education; five major fields which must be involved in the general research framework.

3. The epistemological and phenomenological analysis regarding with educational research must follow a markedly didactical line. The interest must be focussed in obtaining valuable information for teaching and learning, what means thinking about the student, his needs and capacities, about the classroom, its activities and didactical methods. The information gathered through this approach shows the links among the different parts of the two levels mentioned above under a unique reference: *the individual and collective mathematical thought, its evolution, its relationships with other kinds of thought and its education*.

4. This way, the relationships between the Epistemology of Mathematics and the Psychology of Education are in a privileged position, and when focussing all the attention in the processes of creating knowledge it makes sense as a part closely related to mathematical knowledge and to curricular decisions. Likewise, the pedagogical aspect shows a close dependence on the previous factors, to which we should add some other considerations either social, political or cultural that complete a specific and global world view in which multiple relations are needed of a previous integration to carry out the particular studies in the different fields and approaches.

But, such integration must not be finished in a sum of data obtained from different approaches (interdisciplinary conception). On the contrary, it requires a complex elaboration by following a specific research methodology (Didactical analysis) that it must be quite different from the corresponding to the technical part of the process (activities 4 to 10, figure 1). Let´s go to treat next on this argument.

3. Insufficiency of the Interdisciplinary Approach and Specificity

The insufficiency of the interdisciplinary approach regarding Mathematical Sciences Education research, which still has nowadays a wide acceptation by the community although its results can be improved on (Kilpatrick, 1994, pp. 78-79), is based on the relationships among the five major fields and on the nature of the phenomena.

Firstly, the analysis of the relationships may gives new and genuine nuances to the isolated information; this nuances gives cohesion to the study and they provides the integration character we are postulating. Nevertheless, the needs of integration, even importants, not only are justified because of the characteristics of dependencies relations; they can also arise because of the existence of a stagnation of the researches or because of the advanced situation of studies, as it is the case of the concept of function (Harel & Dubinsky, 1992; Romberg, Fennema & Carpenter, 1993) or of the rational numbers (Carpenter, Fennema & Romberg, 1993).

As regards to the nature of the phenomena, we are maintaining the existence of a specific part in the Mathematical Science Education field, to see which it is only needed to analyse the answers to the following questions: Are there important differences between the knowledge of the phenomena in Mathematical Sciences Education when these are analysed from the particular approach of the Didactics of Mathematics and when it is carried out in the context of the general educational phenomena?; Where do such differences lie in?; Is Mathematical Sciences Education a part, in the inclusive sense, of the general field of Education?; If there are any differences, is it correct to use the methods which are usually applied for the non-specific educational research?; Are such methods enough?; Are they priority ones or of preferential application to any other consideration?.

It is not difficult to come to the conclusion that what is characterizing the Mathematical Sciences Education is not the interdisciplinary aspect just like that, but a specific and deeper way of studying the phenomena regarding the teaching and learning processes. This particular point of view can be summarized, in one hand, in the involvement of some specific basic components (the sociocultural as general one and the four remainders as specific and central ones), which have an essential role in the curricular studies (Rico 1997) and among which we emphasize the one that has to do with the *epistemological and phenomenological considerations about the mathematical knowledge in a global framework under didactical purposes*. In the other hand, in the delimitation and analysis of a relationships' network among the four central components (See figure 2) (Gonz·lez et al. (1994); Gonz·lez (1995, 1998)). These two aspects do not intervene in the interdisciplinary approach, which is usually limited to a simple collection of data coming from different approaches.

To sum up: The phenomena regarding the teaching and learning of mathematics show general aspects, which are part of the interest of other disciplines, and specific connotations that introduce differences in the way of approaching the same problems from other fields of knowledge.

Such specificity lies in the important involvement of epistemology and phenomenology of mathematical knowledge in a global framework under educational purposes as well as in its relationships to other fields (pointed out by Vergnaud (1990, p. 22-23)). When taking this into account we can observe an inversion in the assumptions considering Didactics of Mathematics as a specialised branch of General Didactics or of Psychology of Education (Fischbein 1990, pp.6-12).

It is necessary to include a double point of view in the usual research process: a genuine, specific approach, to support and organize the field (activities 3.1, 3.2 and 3.3, figure 1) through a also specific procedure, and a later common interdisciplinary approach to inquiry in the punctual aspects deduced from the previous study (activities 5, 6 and 7, figure 1). Although suitable for particular purposes, the last approach is neither enough nor priority one, but it must be dependent on the results from the previous analysis.

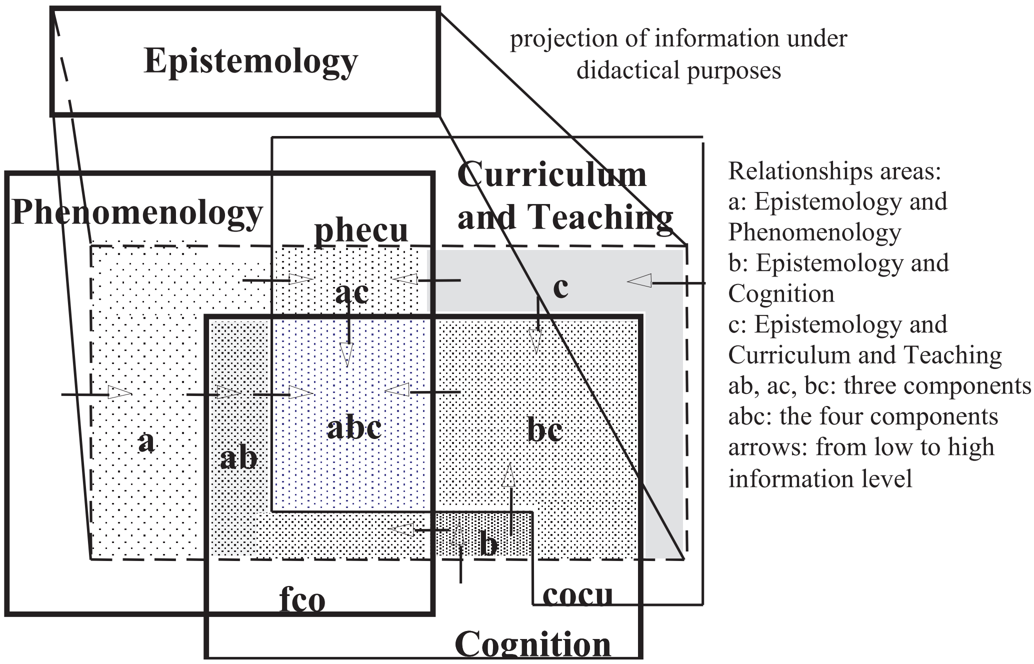

Fig. 2: Relationships' central network

4. Didactical Analysis of a Mathematical Knowledge

In the studies known as *secondary or of synthesis researches* two different methodologies are being used: the *traditional integrating revision* and the quantitative revision, also called *meta-analysis* (Fernández, 1995, p. 165)). Recently, due to the need aroused in many qualitative researches of summarize and integrate a great number of studies, a modality of synthesis, called *multivocal revision* (Ogawa & Malen, 1991) has arisen. It is a procedure of qualitative synthesis "*…aimed at making inquiries on a complex phenomenon of interest in which events can not be manipulated and of which there are many sources of essentially qualitative data, being confident that we can obtain a detailed portrait of the phenomenon under study*". (Fernández 1995, p. 175). The multivocal revision is based on the following criteria, which are similar to the ones suggested for the case studies (p. 176):

1) A clear definition of the research topic by consulting many sources, by keeping evidence's chaines among the records and the inferences drawn and by formally including the informer's reactions to the established conceptual definition.
2) Assess the relative and individual strength of each piece of information by using some of the following criteria: position and certainty of the source (external validity); clearness, detail, consistence and feasibility of the content (internal validity); capacity to corroborate the information through other sources.

Also we are interested in the following criteria regarding the meta-analysis:

3) Revise as many studies as possible; locate them through objective and arguable searches; do not initially exclude studies because of their quality and differentiate and classify each study according to the effect of its results.

The joint consideration of the previous criteria makes up a new approach that we have called *qualitative meta-analysis*. Its finality, as any other meta-analysis, is: "*…the formulation of theories able to explain the phenomena observed in different researches*" (Bisquera 1989, p. 247-252); the difference in this case lies in the use of criteria which are typical from an interpretative approach.

Consequently, we call *didactical analysis* of a specific mathematical topic to the global methodological procedure that integrates and relates, by following a sequential process and according to the criteria of the qualitative meta-analysis, information related to the object of study coming from five basic areas: History and Epistemology, Learning and cognition, Phenomenology, Teaching and curricular studies and Sociocultural aspects. The sequenced process has the following stages:

First stage: Primary revision of the information in every area, following the process in figure 3 and the following steps: a) analysis and classification according to the established criteria; b) gathering the most important data; c) analysis of the relationships among the data; synthesis and conclusions; d) conjectures and research priorities in every area; e) assessment of each one of the area's revision.

Second stage: Analysis of the relationships among central areas according to the diagram in figure 2 and the following process: f) study of the relationships starting from the information of sections c), d) and e) in each area; g) conclusions; h) conjectures and priorities; aspects to research; i) general results and assessment.

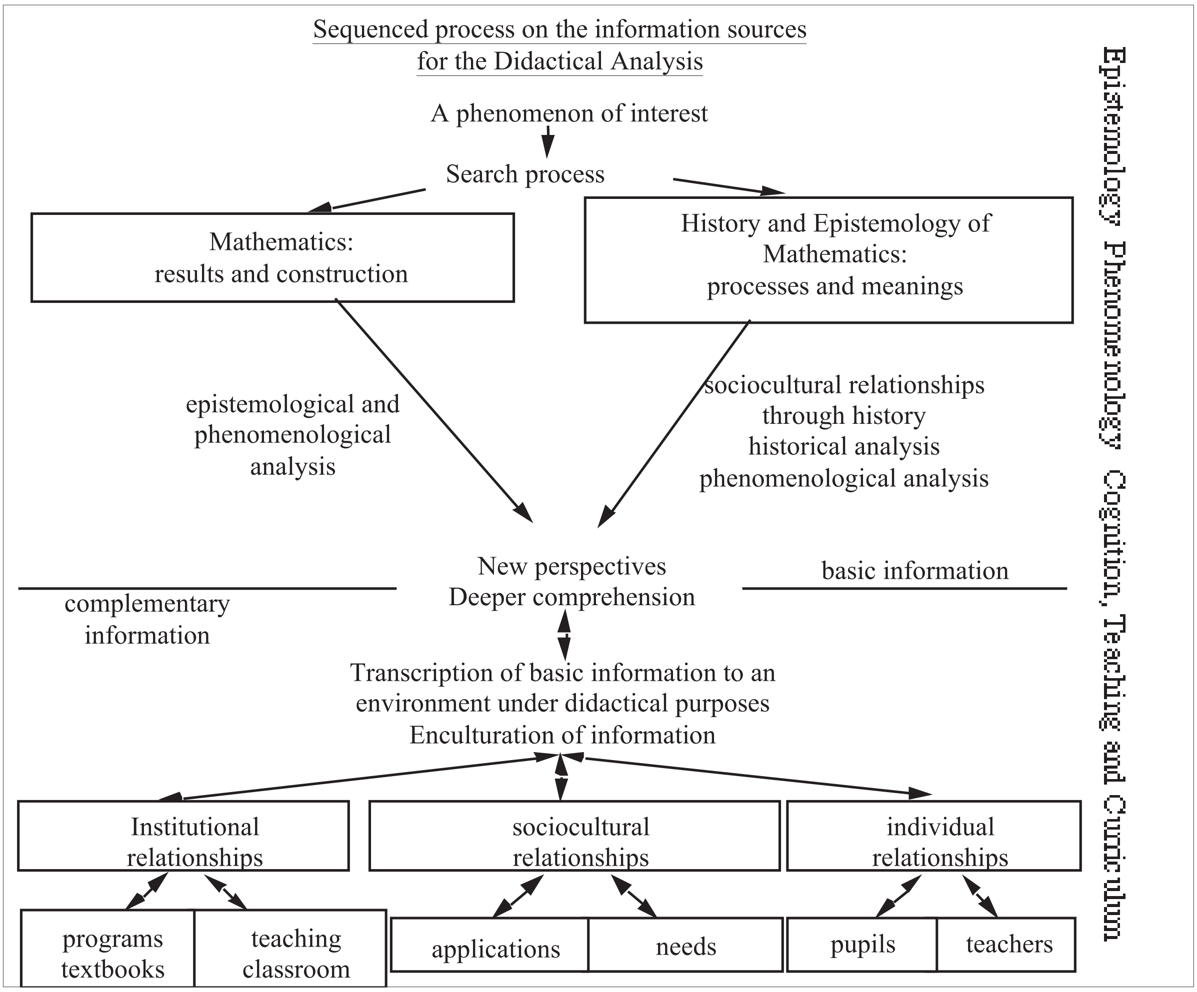

Fig. 3: Integrating information in the first stage of the Didactical Analysis

Figures 2 and 3 describe the basic elements, their position in a sequenced process, the main sources of information and the kind of analysis to be done. Throughout the process data are compared and a global synthesis is carried out. As a consequence, research priorities are established, theories and models are built and empirical and experimental approaches are designed. *The didactical analysis processes, analyses and synthesises information coming from different fields linked to one another by its object of study, giving a synthesis that let the detection of limitations of previous works and properly organizing the future development of the research. The technique used bears in mind the complexity of the field as well as the plurality of approaches and results we can find in the scientific literature.*

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INVESTIGATING TEACHERS' WORK WITH PUPILS' TEXTUAL EIGENPRODUCTIONS

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Abstract: *More and more researchers in mathematics education recommend mathematical writing by pupils as an important activity besides their verbal contribution to classroom communication. A written text in which pupils express their own mathematical ideas in their own language, i. e. by use of words and formulations which are within their individual active language competence and performance, will be called here, according to Selter (1994) a 'textual eigenproduction' (TEP). After a brief characterisation of TEPS and discussing their didactical functions, we present the design and the results of case studies with 16 teachers from Italy and Germany, on how far they are prepared to and experienced in work with TEPS in their mathematics classroom.*
Keywords*: textual eigenproductions, analysis.*

1. Pupils' Textual Eigenproductions

What are textual eigenproductions (TEPs), of what kind can they be and what are their possible didactical functions?

1.1 Characterisation of Textual Eigenproduction

In every day mathematics classroom pupils have rather much to write. However, the main part of their writing is restricted to noting steps of the solution process and results of arithmetical or algebraic tasks, like calculating the value of number terms, transforming terms containing variables, and solving equations. These formalised protocols follow rather exclusively fixed algorithmic procedures and standardised patterns of symbolic representation; in so far we do not call them TEPs. But protocols of problem solving procedures can easily be extended to TEPs if the pupils – using all means of language actively available to them, also formulations of their every day language – produce a detailed description of how they consider the problem content, of what their aim(s), ways, and measures of solution are – there should also be given reasons and justifications for the last ones – and of their results. The TEPs arising from such activity can be called 'commented problem solving protocols' (see Powell & Ramnauth 1992).

Of course, TEPs must not remain restricted to these one type. Other kinds of TEP are:

- Commented problem solving protocols (as described above);
- reports about mathematical investigations (aims, steps and measures, results);
- detailed descriptions and explanations of mathematical concepts or algorithms,
- texts initiated by a specific situation demanding to communicate mathematical facts and relations in written form;
- texts defining mathematical concepts, formulating hypotheses, arguments or proofs in relation to a mathematical theorem.

Writing in the meaning of TEP may happen from time to time in the mathematics classroom; it also can become a regular pupils' activity. Waywood (1992), e. g., reports about journal writing, Gallin & Ruf (1993) call the texts their pupils produce regularly 'journey diaries'; the writing deal with 'core (mathematical) ideas' on which they are expected to reflect, and which they are wanted to invent.

1.2 Didactical Function of Textual Eigenproduction

There are some reasons why TEPs should be introduced into pupils' mathematical classroom work:

- TEP stimulates the individual pupil to analyse and to reflect on mathematical concepts, relations operations and procedures, investigations and problem solving processes he/she is dealing with. Thus, he/she can arrive at more consciousness, and a deeper mathematical understanding of them.
- TEP is able to improve pupils competence and performance in technical language, since it leaves them time for a careful and reflected selection of language means, and thus encourages them to make an active use of technical terms and symbols (see Maier 1989a, 1989b, 1993 and Maier/ Schweiger 1998).

- TEP gives the individual pupil a chance to get control about his/her understanding of mathematical issues by means of a reasonable and reflected feed back from the teacher's and other pupils' side.
- TEP enables the teacher to assess previously and actually constructed knowledge and understanding of mathematical ideas in a more detailed and a deeper way than it would be possible on the base of common written tests, normally carried out in the manner of non-commented problem solving protocols.

The use of TEP in the classroom and moreover the positive effect on the learning process and its evaluation mentioned above are certainly not a matter of course. There are many preconditions for success; we restrict ourselves here to mention two which appear us of high importance:

- If pupils are wanted to produce texts which can give deep insight into their ways of mathematical acting, thinking and understanding it has to be made sure that they address their TEPs to someone who needs full information about the matter written about. Usually they tend to imagine the teacher as the only addressee of their writing, whom they insinuate already to know all they have to communicate and just wants to examine their ability of coming up to his quite specific expectations. Thus, they feel no need for a detailed and explicit description and explanation. Motivations suitable to change the writers attitude to a role different from the pupil's – e. g., there may be used instruction like „Imagine you were a father/mother, a teacher, ...“ (see D'Amore & Sandri 1996 and D'Amore & Giovannoni 1997) –, to write (a letter) to a younger child or a classmate who missed classrooms for reason of illness and should be informed about what has been learnt in his absence, to write a diary, to design a poster for an exhibition, to formulate a lexicon article, etc. Or the teacher can organise a particular communicative situation in which, e. g. a pupil has to describe a geometrical drawing in a manner that a classmate is able to reproduce the figure only on the basis of this description. Sometimes it may be helpful to defamiliarise the common problem solving situation by means of open or incomplete tasks (see D'Amore & Sandri 1997, 1998).
- The teacher does not only need good ideas for an effective stimulation of the pupils, but also for an adequate way of working with the produced texts. Above all he has to be prepared and able to interpret and to analyse the TEPs carefully

and competently. It is, indeed, not easy to find out the mathematical concepts, thoughts and ideas which are on the base of the particular text produced. It needs not only high attention but also a lot of experience and possibly some training as well.

The main research interest of the project the design and results of which we are reporting subsequently was on the second precondition. In how far do present teachers make use of TEP and how experienced are they in their interpretation and analysis? Are they convinced that TEP could be a help for their mathematics teaching, mainly for evaluating individual pupils mathematical knowledge and achievement? In detail the questions of research were as follow:

(1) Which instruments do teachers usually apply for assessing pupils in mathematics? What is in the centre of their evaluation (knowledge or abilities), and in how far and in what way is their assessment individually or collectively oriented?

(2) Do they know further instruments of assessment (they do not use themselves), and what is their attitude towards them? Do they, in particular, at least know tep as a possible instrument of assessment, and which is their attitude towards it?

(3) How do they interpret and analyse TEPs? Which is their particular point of view and the effectiveness and competence of their interpretation?

(4) Is a first experience with the analysis of TEPs able to modify the teachers' attitude towards them as an instrument of assessment (and in what way)?

2. Teachers Interpretation of Textual Eigenproductions

In this paragraph we describe the design of our investigation before we are going to report about results. The investigation is methodologically designed according to the rules of an interpretative research paradigm and qualitative research methods (see Beck & Maier 1993, 1996, Glaser & Strauss 1967).

2.1 Design of the Research

We carried out a sequence of three interviews with each of 16 mathematics teachers from Germany and Italy.

In the first interview the teachers were requested to report about what is in the centre of their attention when they assess their pupils' mathematical competence in the classroom, and which are the means they usually apply. In addition they were wanted to tell about their knowledge of assessment procedures not applied by themselves, and about eventual reasons for not making use of them. In case TEP was not mentioned until then, the teachers were asked if they know it, which were their possible experiences with it, resp. what do they imagine under this label and did they ever consider its introduction into their mathematics classroom. The interviewing person tried to put all these question in an open manner, to react in a flexible way to the teachers' utterances and to put more exact questions in case of unclear or insufficient replies.

Afterwards every teacher was presented TEPs which have been produced by (Italian) pupils unknown to him/her, as an answer to e. g. the following instruction (about the origin of these texts, see D'Amore & Sandri 1997, 1998): *Imagine you were a father (a mother) .... Your young child, 7 years old, learnt from somewhere that every triangle has three heights and asks you: „Dad (Mum) what does that mean ?" Nothing is more intriguing than leaving young children's questions unanswered; therefore, you decide to give the following reply: ...*

The teacher was given texts produced by the pupils e.°g. the one written by Simona (Scuola Media 2; 11 to 12 years of age) saying: „*My son, yet you don't know geometry, but I will explain you what height means. Like you I and papa have a height which is measured from the head to the feet; the triangles have one as well, but their height is measured from the vertex, which is a little point, down to the base, which is like our feet. Since they have got three little points (vertices), they have three heights because they have three pairs of our feet. And since we have only one head and one single pair of feet we have only one height.*"

Having received these TEPs the teacher was requested to read and analyse them carefully until next day, in order to find out as much as possible about the writers' mathematical knowledge and competence in reference to the respective topic.

In the second interview every teacher gave his interpretation of the TEPs he/she had been given the day before. He/she was expected to make comments on how far the pupils did understand the task and how he/she estimates their answer, to describe eventual error strategies or wrong thinking on the pupils side, and finally to give a judgement about the pupil's ability. Afterwards he was wanted to consent to the plan of letting his/her own pupils write texts initiated by the same instructions the Italian pupils had been administered.

Some of his/her pupils teps, carefully selected mainly after the criterion of diversity, were handed in to the teacher in a typed form and he was asked to be prepared for analysing them like the Italian texts a third interview.

In the third interview the teacher started to talk about his/her general impression on the pupils teps, before he/she entered into a detailed interpretation of one text after the other. At first he/she could not be certain about the respective author and, thus, was able to analyse it without making use of previous judgement on the pupils mathematical knowledge and ability. Afterwards, he/she tried to guess the respective writer's name and was then informed about it. Knowing the name, he/she could make additional comments, mainly on the question of if he thinks to have learnt something about this individual pupil he/she did not know before, and what.

Finally the third interview turned back to a general discussion about tep. In case the project gave the teacher the first opportunity of experiencing it, the interviewer wanted to know what he/she thinks about it as a means for getting information about individual pupils mathematical competence and ability. In case he/she already knew tep before, the questions were directed towards possible change in his/her attitude towards it. In any case the interviewer wanted to know if the teacher intends to make use of tep in his future mathematics education.

All three interviews with every teacher were audio-taped and transcribed for further analysis. All together, they formed the data base for one case study on the research questions mentioned above. The investigation started with two cases, which were immediately analysed, in order to formulate, by way of comparison hypotheses and theory elements. Afterwards the next two cases were selected and analysed for means of further comparison and further development of the theory (theoretical sampling in the meaning of Glaser & Strauss 1967). In that way, so far 13 cases were studied with teachers as follow:

- 8 German secondary school teachers, 2 of grade 6 (pupils of age: 12), 1 of grade 7 (pupils of age: 13), 2 of grade 8 (pupils of age: 14) and 3 grade 9 (pupils of age: 15)

- 8 Italian teachers, 1 from „biennio superiore“ (pupils of age 14 to 16), 5 from „scuola media“ (pupils of age 11 to 14), and 2 from „scuola elementare“ (pupils of age 6 to 11)

2.2 Some Results

At first we report about the teachers' description of the criteria and the means of their procedures of assessment in the first interview, and how they interpreted Simona's TEP in the second interview. Then we report about the teachers interpretation of their own pupils' TEPs in the third interview and their concluding comments on tep as an instrument of assessment in their classroom.

Criteria and means of assessment
Asked about the means they use and the criteria they are oriented at when they are going to assess their pupils' individual achievement in mathematics, many teachers emphasised to apply several methods, but most of them – Italian as well as German teachers – referred to the same, namely:

- Direct questioning or verbal examination in the classroom;
- small written tests (referring to a small topic area), in most cases without announcement in advance;
- bigger written tests (referring to a larger topic area) of a lessons time at least, in most cases announced in advance

Typical statements are as follow:

- Italian teacher D: „*No, there is not a single method I use. We can say that I often administer verbal or written examinations, without announcement as well. From that I have always many information*.“
- Italian teacher A: „*I use the classical methods: verbal and written examinations, small tests*.“ – „*From time to time I also administer written tests without announcement. Nevertheless, most times the pupils know, they expect, they know it*.“
- Italian teacher B: „*No, ... what is the matter? I ask the pupils many questions*

> *directly in the classroom. They know that and answer. I like talking to them. This is a nice report given in trust. In case they did not understand they themselves ask me. And then, every four weeks I administer a written examination and also small tests*."

These common means of assessment can be characterised as formal. In most cases the pupils have to answer to direct questions or to solve standard problems. Thus their answers resp. their partial or final results can be judged as right or wrong, according to norms fixed by the teacher. Thus, the outcome of the assessment can be used for a pretended possible ordering of all pupils of the class in behalf of their individual level of achievement. The order can even be sectioned into a few steps, represented by marks of in between 1 and 5 or 6. It is, briefly said, a normative evaluation of pupils achievement, the main criterion of which seems to be a correct and quick reproduction of memorised definitions and algorithms.

But there are a few exceptions. The German teacher K. points out as his main instrument of assessment the observation of individual pupils, which seem him able to „*make weaknesses evident quickly, and can be released individually*". However his observations are restricted to verbal contributions and to results of individual, partner or group work in the classroom.

The German teacher S. also directs his attention to observation in classroom. He organises individual work, in which the pupils work independently and he „*can walk through the rows and see, what kind of difficulties some pupils have, and help to overcome them*". In contrary to K. he also demands individual pupils to work at the blackboard and organises classroom periods in which the pupils have to work independently.

The German teacher We sets up, among other means of assessments, brain-storming at the beginning of every classroom. There, the pupils tell all they know about a given topic. That way the teacher gets at least a kind of collective control on their competence. His intention is „*to get a holistic impression of pupils' knowledge about facts and formulas and about their ability to apply them in adequate situation as well*".

The German teacher K applies a really extraordinary mean of assessment: „*I* *provoke the pupils by very difficult or nonsense problems, which they try to solve in groups. The group is intended to succeed in finding a meaningful solution, which is realised as soon as all group members one after the other, also the slow learners among them, is able to explain the solution to another pupil, who was not able to find a solution*."

The question if the teachers know means of assessment which, for some reasons, they do not use themselves brought little concrete answers. Some teachers simply confess not to know further ones. Others restricted themselves to say there are several ones, but they all are not applicable. Nobody mentioned or even described such means. Typical is the utterance of Italian teacher B: „*Of course I know more than one, but I am not sure, what you want?*" Only one teacher mentioned TEP spontaneously in this part of the interview, this is the German teacher S.

The German teacher S calls tep a „*mathematical composition*". There are some reasons why he does not apply it: „*Mainly because of many pupils from foreign countries with heavy language difficulties I can not expect satisfying results. In addition this kind of assessment demands an enormous expense of time, not only in the classroom but also for revision*." Nevertheless, he thinks its use meaningful in a class of sufficient language skills.

Directly addressed with the question if he knows TEP, the German teacher We said he had some experience with TEP and thinks it to be „*an adequate mean of assessment*", but for himself he restricts it to the topic of equations. In this case the use of TEP seems him „particularly adequate, since due to the production of text the pupil has to build up the logic of the task and he/she needs a fundamental understanding". However, he wants to distinguish between assessing and evaluating (marking) pupils achievement, since he never would tep use for the latter one.

Similar utterances made the German teacher Gr who teaches at the same school as teacher We. He also let pupils write teps which deal with solving equations, and he admits that, so far, he „*never let the pupils write on geometrical questions*", and that he has „*difficulties to imagine how that could look like*".

The German teacher Wö claims to know TEP and says „*that he could well imagine to* *use it as a means of assessment*." But in case of his present class he „*would have little hope to come to a reasonable result*". The other German and all Italian teachers confessed not to know TEP.

Teachers interpretation of Simona's TEP

By two German teachers we received quite interesting analyses of Simona's TEP. Therefore, they shall be quoted in full extension:

- Teacher Gr: „*I think up to this point it is not bad: '... since the height is measured from the vertex and since the triangle has three vertices there is one height possible from every vertex to the baseline' up to the statements about the three pairs of feet. From that point on I can no longer follow her.... well, she should have said that the height is perpendicular to the baseline or something like that, but I think this could be clarified in a talk. They do not just think of everything, while having to formulate. However I think it could be said that in fact she did understand a bit.... I also thought about how I myself had answered the question, but even I would not be able to explain it to a seven year old child in a manner that it would enable it to understand. For certain I could explain it mathematically but in a other way probably with help of a plummet or so. It would have to be explained rather manually, otherwise a seven year old child will not understand. So to say there should be presented a real triangle and made fall down a plummet, turn the triangle and make fall the plummet again and so on etc. That way it can be seen that it always goes down vertically. But certainly nothing can be achieved with words like 'perpendicular'. I can tell you quite seriously that I myself had some difficulty with this task. Therefore, I think that probably many pupils go back to what they learnt in the classroom and less enter the question how it could be explained to a child*."
- Teacher S: „*Yes, I have to say, not bad. If the child really is of age seven the answer fits to a certain extent. In parts the answer is not bad; I would only say, that the father or the mother could provide an illustration for the child. And about the height: is it allowed to say we have one height and the triangle has three? I would see that rather a bit different. I would not make such strong link to the height of man. Okay, it can be said, there is some relation, but the link of the one height of man and the three heights of the triangle is not so good. But in principal, for a seven year old child such an explanation, why not. It should be*

said in addition how this height is to be located, that it is the shortest tie from exactly this point to the opposite line. This should be included."

These analyses, given by the two teachers who already had some experience with TEP, are untypical for at least the following reasons:

- Going into the details of Simona's text, they try to give a complete description of her concept and ideas about height indicated in the text, and they compare them with their own – as the firs text shows, possibly questionable – mathematical knowledge about this issue.
- As far as they evaluate the text they do it in a differentiating way.

This is different in the case of the following statements:

Italian teacher B: „*Oh yes I have been amused. This answer hits the topic well, an answer of high level, since she is able to explain this thing to child of 7 years, that means that in her mind she has perfectly understood what the heights of a triangle are... You have also to see how she tries her best to say it in a good manner*.

German teacher We.: „*I think the pupil did fundamentally understand the problem. She explains this in a nicely visual way and in a manner that someone without previous knowledge could get a good imagination. I would be happy about this answer*."

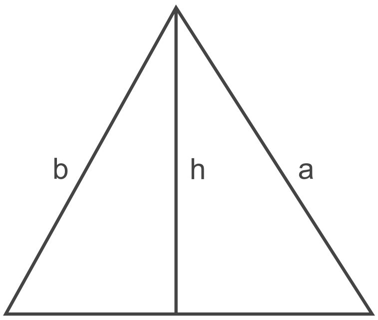
German teacher K.: „*I find this Simona excellent. The answer has to be read carefully: 'A triangle has three tops.' And in a pupils mind normally the triangle is normally situated like that: 'The point on top the feet at the bottom.' She simply turns the triangle around. A triangle has three heads, compared with man. In case this is really a child, it must be of really high intelligence. In relation to a seven year old child the question is actually completely answered. Hits the point without blab. Very visual. I am quite exited*."

Teachers' interpretations of their own pupils teps

The interpretations typically given by nearly all teachers in reference to Simona's TEP and to the TEPs of their own children – in case of Italian and German teachers alike – hardly differed from each other.

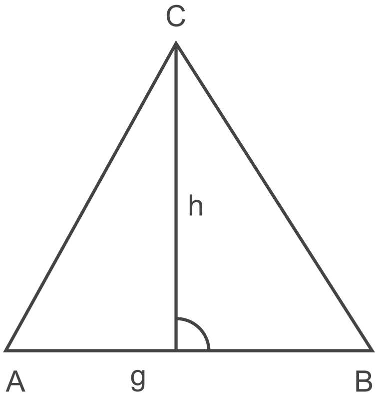
They shall be characterised at hand of five TEP examples as follow:

(I)



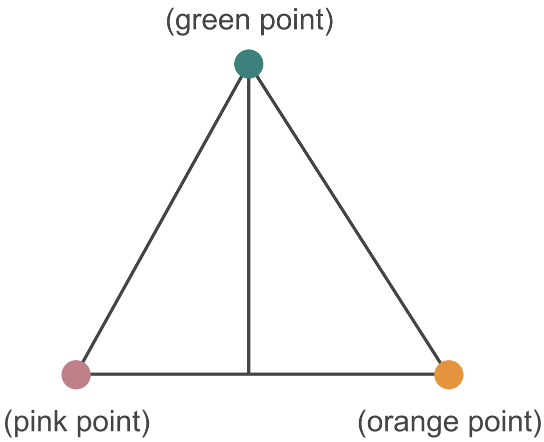
„The triangle has only one height and a baseline by means of which area and circumference can be calculated.“

(II)



“A triangle has always one height and this is always from the baseline to the point C. The height always starts perpendicularly from the baseline to the point C.”

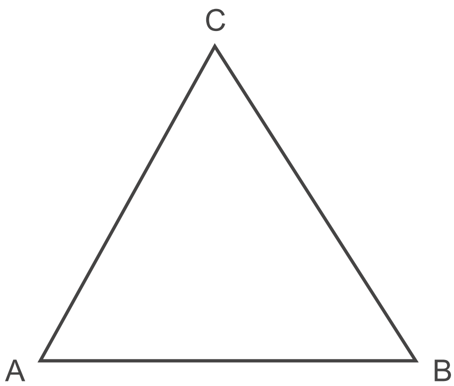
(III)



„Here I’ve drawn a triangle and when I now draw a line from the green corner downwards to the straight line this is called height. When I now draw a line from the orange corner upwards and from the green corner draw a line to the side the line upwards is also called height. When I do the same from the pink corner and from the green corner as well, the line upwards is again called height.“

(IV) *„Triangle is a plain figure, this triangle has one height, and since the triangle can be turned around it consequently has three heights.“*

(V)



„The first height goes through the vertex C. It stands straight on the opposite segment. The second height goes through the vertex B. It stands straight on the opposite segment. The third height goes through the vertex A. It stands straight on the opposite segment.“

a) Evaluating, not descriptive interpretation: *Above all the teachers' interpretations were strongly characterised by evaluating more than describing utterances. The teachers seemed not so much interested in what the individual pupil really thought about the topic in question, i. e. in the mathematical concepts and mathematical knowledge indicated by the text. They rather draw direct conclusions on the pupils general level of achievement. Some German teachers even felt able to fix this level by attributing exact marks to the text author. For example* TEP *(I) was commented by the author's teacher as follows*: „Perhaps I would say this is a pupil from the middle field. That about baseline and the height for the calculating the area is correct, but it is not asked for the circumference. Here he has mixed up something." After being told the pupils name: „Yes, he really tends more to the better ones; so to say he tends to two."

TEP (IV) was very briefly commented by the teacher, saying: „*Since one is able to turn the triangle around. Well, I think, probably, he may have thought of the right thing, any way, because the turning seems to be a kind of correct approach*."

What the teacher not realised was that pupil who produced TEP (I) evidently regards lines in the triangle not as autonomous (geometrical) objects. The words „baseline" and „height" represent in his mind magnitudes which are needed to calculate the triangle (area and circumference). That means that he interprets them functionally under an elaborative aspect. The triangle „has" the baseline and the height so to say exactly for that reason. Thus, it has to be seen as consequent not to make further differences between both. In calculation both have the same function; there is no need for differentiation. That goes so far that the pupil evidently uses the labels „baseline" and „height" synonymously, regards the baseline as the second height and talks about a „third height" although so far he had only mentioned one. The instruction evoked in the pupils mind evidently rather an algebraic than a geometrical imagination. Even the word „triangle" is used by him less as a name of a geometrical object than as a stimulus for selecting a certain algebraic procedure.

This is quiet different in case of text (IV) in an expressively conceptual approach its author calls the triangle a „plane figure"; i. e. she attributes it conceptually to the category of plane geometrical figures (polygons). The pupil also shows a quite clear intermodal conceptual imagination about height; in her opinion this is a line which only exists in a triangle one side of which is horizontal, i.e. parallel to the top and bottom margin of the sheet. This is since the height has to be vertical, i. e. parallel to the right and left margins of the sheet. In spite of this quiet narrow concept of height the text author succeeds to give a reason for the triangle having three heights. She imagines to turn the triangle in the plane so that one after another each of the three sides once becomes horizontal. That way it has a height for each of these sides, what means in total three heights. (It seems interesting that in this case a line of a triangle can loose its property to be a height when its position changes. On the other hand the line gets this property back at any time when the particular position is restored. This is sufficient to make the existence of three heights argumentatively certain.

Possibly the teachers followed their habit to use written products from the pupils' side for assessing their mathematical achievement according to class internal norms. Thereby they did not so much follow what is called a central tendency. Rather they tended to evaluate the text (in reality the pupils) in a strongly dichotomic way. They appeared to them as either totally good or absolutely inadequate or bad.

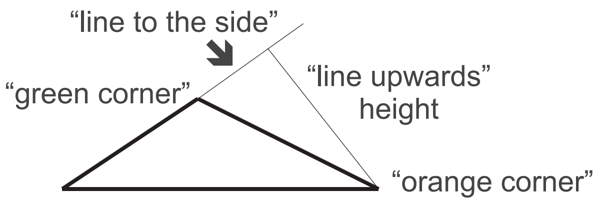
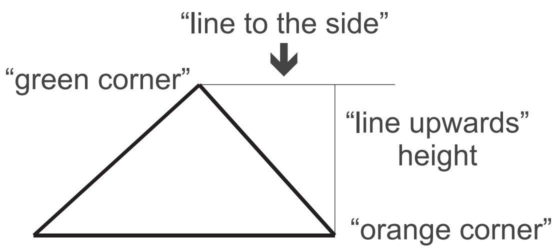
The criteria on which the teachers based their valuation were mainly about

- how much a pupil came up to the norms of a formal mathematical language (in many cases, correct mathematical statements given in the pupils own language, were not accepted),
- how near the pupils utterances were to the own teachers content and way of presentation in the classroom; they were seen more positively in case they showed similarity with contents presented or the language used by the teacher, and original ideas from the pupil side were, at the most, classified as „interesting", but not highly appreciated.)

For the criterion mention last the interpretation of TEP (II) by the teacher gives an interesting example: „*This answer possibly has already something to do with the Pythagorean Theorem. Since we already have started with it, and, therefore, it may be that this plays a role in the answer. With the perpendicular line where we then sometimes have calculated the baseline. This is possible since we particularly went down starting from the vertex C. I would rather attribute this to a good pupil*." And confidently when told the pupil's name: „*Yes, Stefan has a two*."

b) Unequivocal, not open interpretation: *As a surprise, evidently the texts appeared unequivocal to nearly all of the teachers. They felt able and showed no doubt to decide exactly about the authors' thinking and even about their particular level of achievement. None of them generated more than one particular interpretation about the author's mathematical concepts and ideas*. Example:

In text (III) the height which goes from the „green" point to the „straight line" (means the horizontal drawn baseline) is relatively clearly to be identified. But how shell we interpret the following sentence „*When I now draw a line from the orange corner upwards, and from the green corner draw a line to the side the line upwards is also called height*." There are at least two interpretations which seem of the same probability which can be made clear by the following drawings.

The pupil imagines a height line outside the triangle, perpendicular to an elongation of the opposite side.	
The pupil has a height line in this mind which is parallel to the right and left margin of the sheet and runs towards a parallel to the (horizontal) baseline through the „green" edge.	

The teacher, however gave a very simply analysis like that: „*From the formulation it could be Johann B. he is between two and three and tends to make careless errors. He does formulate it completely, I am rather pleased with it, since I myself work much with coloured chalks at the blackboard. The drawings even become more understandable you can better realise the relations. Really, I like that rather well*."

c) Global, not detailed interpretation: *It could be observed that the teachers did not analyse the texts word by word, sentence by sentence in a detailed manner. Thus, they were neither able to differentiate their valuation and to relate it to the mathematical* *content, nor to describe the pupils mental processes resulting in their text in detail and with sufficient exactness. In most cases the teachers analyses can be regarded as superficial and incomplete.* TEP *(V), for example, got this interpretation*: „I think that pupil understood the task with the heights. He fixes the height exactly through vertex C and through the opposite side; the other two as well. Certainly he does not say that they have to be perpendicular to the baseline. But the answer is not that bad.“

In this text, indeed the teacher picked up a lot. But he could have mentioned that the pupil does not say „*through the opposite side*“ but describes that each height „*stands straight on the opposite segment*“ (German: „*steht gerade auf der gegenüberliegenden Strecke*“). This can be seen as an exact equivalent from every day language for the mathematical expression „is perpendicular to the opposite side“. In addition the text author talks about „*the first*“, „*the second*“ and the „*third height*“. This is his way of an indirect proof of the statement: „The triangle has three heights“.

In text (I) it should have been taken into account that the text describes „*triangle*“ and „*height*“, talks about „*baseline*“, „*area*“ and „*circumference*“ and that it also mentions a „*third height*“, although until then it had only be talked about a single one. In addition the author points out that he does not know anything about this third height. Also TEP (III) speaks in the I-form. In contrary to (I) it relates excessively to a added drawing in which the edges of the triangle are marked in colours. The pupil talks about actions of drawing, where certain directions are attributed to lines: „*downwards*“, „*upwards*“, „*to the side*“, „*towards the strait line*“. And it forms if-then-sentences: „*If I draw..., this upwards running line is called height*.“

Teachers' final comments on TEP

Many teachers were surprised when they compared their evaluation of the TEPs with their previous meaning about the author. Many found excuses for „good pupils“ having written „bad“ texts and „bad pupils“ having written „good“ texts and did not get the impression that TEP could help them to learn more about their pupils mathematical competence and thinking. Few, however, ended up with exactly that conviction and promised to use the instrument regularly in the future.

Consequences

If the didactic function of teps, described under 1.2, ought to become reality, it seems absolutely necessary that the teacher makes use of them in an adequate manner. This means that he provides regular training for the pupils in text writing, and that he is prepared and able to interpret and analyse the texts in a descriptive rather than evaluating way, in detail and completely and not in quite a general and selective manner, conscious that every text is open for different interpretation. Evidently this ability is not a matter of course; it has to be learned and has to be trained.

In addition the teacher has to be convinced that his teaching and his organisation of learning processes can have, and normally has different effects on the individual pupils side. These differences are not only quantitative ones, i. e. different pupils pick up more or less from the teachers offers, but are in the same way of deeply qualitative kind. This means that the pupils construct qualitatively different concepts, knowledge and mathematical thinking. The teacher should be eager to learn about this differences and to draw consequences from it for the planning of his classrooms.

Making pupils write TEPs and appreciating this activity as a stimulus for learning and a means of assessment has to be based on a particular philosophy of mathematics and mathematics learning. It demands a particular opinion about what it means for the pupils to *do* mathematics.

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RECENT TRENDS IN THE THEORY OF DIDACTICAL SITUATIONS

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Abstract: *During the past thirty years the theory of didactical situations developed by Guy Brousseau and other people was at the centre of the identifications of the main characteristics of didactical systems as far as they are set up to teach mathematics. Following the great trends of the theory in its historical and notional development, from the first modelisation of the mathematization process to the identification of the crucial role of institutionalization, we focus our attention on the more recent questions calling for evolutions as new fields for research: regulation of didactical systems, organization of sets of problems for teaching, relationship between teaching devices and concepts of the theory in various contextual environments such as teachers training and the use of so called new technologies of information.*
Keywords: *didactical situations, institutionalization, regulation.*

1. Introduction

The relations between teaching and learning are at the centre of concerns for all of those involved in the field of mathematical education, as much through the everyday action in the class as for an action within less immediate range which leads on to the construction of a curriculum proposal.

Il is also a central matter for research, which could plan as a reasonable purpose to provide one or several theories giving an account of associated phenomenons. From the famous Socratic dialogue of Menon to the most recent models as the Bereiter one (1985), we all know many ways to give an account of the fundamental paradox of teaching-learning. We are going to develop briefly here one of the most powerful theories for the study of this matter in the mathematical field, the theory of didactical situations. It was built on the identification of the founding role of the didactical intentionality and on its realization through the a-didactical/didactical dialectic. We will bring to the fore the central role of institutionalization in this process and we will try to identify how the theory of didactical situations allows to undertake the study of matters the teaching of mathematics comes up again.

2. The Classical Components of the Theory of Didactical Situations

At present we know two theories which get organized on the identification of facts and phenomenons of a didactical sort. These two theories, the one of didactical situations and the anthropological theory, are structured on the following fundamental principles: the mode of social realization of the didactical intentionality towards a learning of knowledge, social practices or techniques considered as stakes, are inseparable from the epistemological and functional features of these knowledge.

Studying the different forms these modes of realization take in the context of a personal plan or in the context of a social one - which is the case of mathematics -, studying the restraints they have to deal with, the variables structuring them and determining the space for the didactic action as well as the role, places and responsibilities of people acting, teachers and pupils, represents the central object of research in didactics.

The production and the construction of mathematical knowledge, as well as their transmission, follow equivalent structuration roles. The mathematical knowledge, in the broadest sense of the term, cannot do without didactics, as much in its development as in the forms through which its transmission and its teaching are achieved. Therefore we can consider that there is an autonomy and a specificity of the didactics, especially concerning mathematics. The psychodynamic process from Dienes, proposed in the beginning of the sixties as a pattern for the construction of mathematical knowledge showed a certain amount of restrictions which did not allow him to suitably convey characteristics of the didactical interactions. It’s greatly to Guy Brousseau’s credit that he proposed another modelling allowing to overcome the difficulties of the former pattern. The basic idea was this : a mathematical expression process consists of several steps and is based on specific games for which objects are brought into play, these objects being necessary for the realization of this process and being objects of knowledge : game of interaction with an environment with regards to an action, game of interaction with an environment especially structured for the communication with regards to formulation, game of interaction with an opponent (which may include the adjustment of decision rules) to sort out statements from their trueness character and to determine those having a trueness character from those definitely not having it. The study of basic situations corresponding to the dimensions of action, formulation and validation produced significant work and allowed in certain domains, wether to let some questions come into the didactic field whereas they did not seem to be relevant through look of a suited problematic, or to renew the approach of certain problems. Let’s quote particularly and respectively the research work of J. Perez (1983) about the creation of a code at kindergarten, the work of N. Balacheff (1988) about proof and demonstration, and the numerous situations G. Brousseau and his students built and proposed in the context of the COREM. Hence the theory of situations formed itself into a theory of learning derived notably from the cognitive psychology, the one of learning by adaptation. In this respect, it’s important to notice that the transfer made by the theory of situations has got as a result (as much as an object) to move the place of questioning from learning itself to the conditions of learning; Then it will be possible to come up to the theorization of the teaching facts without the prerequisite of a theory of learning.

Let’s resume these ideas to go further on. These are ideas structuring the thematization peculiar to the theory of situations. Mathematics is the product of specific games having dimensions of action, information and veracity. To learn mathematics, one has to meet these games, hence to practice them. These basic games exist in the familiar environment of any individual or young child. Some of them are timely organized by adults, some are not. The later ones are called no-didactical situations, the former ones are called didactical situations (or with didactical components). Unfortunately, in most of the cases, what is at stake in these situations is not the mathematical objects or knowledge. Only the elementary numerical knowledge as the one of the chain of numbers are acquired in this way, especially in the family circle. The acquisition of more sophisticated mathematical knowledge can only be very rarely produced by circumstances when no didactic intentionality external to the individual appear. In other respects, the situations are not all equivalent as regards to the way a knowledge is brought into play, and they are even less equivalent when it is a matter of taking this knowledge at stake. The assumption is made that some of these situations have got a special generic value, these are fundamental situations. The determination of fundamental situations of mathematical knowledge is a central issue for the research in didactics. Two examples especially emblematic of situations having the required characteristics are to be found in Brousseau (1997). The effective realization of these situations, or of situations possessing rather good characteristics in a school context, introduces the non didactic in the heart of the didactics or of a didactical project. In that case, we can speak of an a-didactical situation. The a-didactical situations, as regards to their conception, to the realization and to their leading (by the teacher), involve a high level of requirement. Actually, within their principle itself comes the ability (their ability) to allows the appearance, in a determined form, of a new knowledge, only by the interplay of interactions led by the situation and without any direct didactical pressure from the teacher.

3. Knowledge and Institutions, the Particular Case of Mathematics

Mathematics would not be the product of a spontaneous generation. Actually, some of their ways of existence depend on the social devices and on the sets of practices in which they fit. There is a form of universality of the mathematical theories, and this up to a certain point, as showed by the existence of the intuitionist program. However, techniques themselves are not always universal. We actually know, and the ethnomathematics work confirms it, that stable and relatively specific sets of mathematical techniques exist, and spread in organizations and/or fields of practices for which they fit needs. M. Douglas (1985) especially studied this phenomenon in "*How institutions think*". Then we will generically define as an institution any specific social grouping, operating on the symbolic mode and for which a knowledge is at the principle of its constitution. Hence institutions and knowledge (of the institution) co-define each others. The mathematical objects, the mutual relations they keep up, their modes of designation, the calculations associated to them, the problems and questions they allow the study of, appear in numerous institutions, from home economics to astronomical calculations, or the one of mathematical science itself, the one of professionals. Each of these institutions has got specific needs, and must therefore learn the mathematics useful to it, as well as for any people entering the institution, hence wishing to have, with the other people from the institution, relations consisting in having common mathematical practices, the ones of the institutions. From the point of view of the transmission of mathematical knowledge, we will consider as phenomenons of the same hand the ones concerning the individuals and the ones concerning the institutions.

The relations an individual, a group of individuals or an institution has with mathematics, relations updated in practices where mathematics can be observed (techniques, statements, forms of proofs) are not always explicitly pinpointed by their actors as knowledge. In this way, one can calculate areas without being able to express some properties of the notion of area (not having studied it or not having to study it). One can use mathematical techniques enabling to solve division problems by calculating a quotient and a remainder without to be in a position to identify these techniques as being a matter for division. These techniques, even of they have been learned, wether by mutual contact, wether in a non didactical or a didactical context, appear as knowledge. They will be converted in learning by institutionalization, becoming autonomous from the conditions of their emergence, de-contextualizad. “Les instruments culturels de reconnaissance et d’organisation des connaissances sont des savoirs, objets d’une activité spécifique des institutions, ou d’une activité d’institutions spécifiques”, G. Brousseau (1995). The mathematicians’ community, by its permanent work of reorganization of mathematics, and especially the most recent mathematics, performs institutionalization.

A didactic theory of institutionalization has been studied by Rouchier (1991). A general outline of institutionalization has been brought out by action (of institutionalization) on an “acted” situation, in which knowledge mainly as knowledge in action (situated knowledge) have been involved by the subjects. These results led to the start up three series of work. The first ones are directed towards the explication and the realization of this outline in specific contexts, relating to the objects and stakes of knowledge connected to these objects. The research on the didactic memory of teachers (J. Centeno (1995), the studies related to the structuration of the milieu (C. Margolinas) can be linked with this first series.

The second ones are directed towards the study of the relations between the notion of condensation as proposed by Arzarello, Bazzini and Chiappini (quoted by T. Assude (1994)) and institutionalization work from T. Assude (1995), especially from the angle of a-didactic theory about conceptualization.

The third ones are directed towards the modes of the actual realization of this transformation of the "acted" situation and the ways students and teachers are involved as " cognitive" actors of the whole process. Among the various studies that were done in the field, the work of G. Sensevy (1994) is one of the more spectacular. He put in evidence, in the field of teaching-learning of fractions in the elementary school, the crucial role of students' writing about their own knowledge and the debates these writing permit to set up. Theses debates are centred about crucial questions such as : "*What can be the signification of a fraction greater than one ?*" coming from the students and discussed by the whole class.

4. New Objects, New Problems

We are going to present now three series of questions, of problems being the concern of the didactic sphere and calling for new developments of the theory of situations. As questions, they are not rigorously truly new, but a matter of fact the actuality of some of them is modified, and moreover we have theoretical and experimental means at our disposal to suitably problematize them.

The first series of questions is associated to the stability of the didactical system. We can actually express as a general principle, due as much to necessity (in the sense of being necessary) as to the observation of its empirical nature, that any didactic system cares about upholding its stability. We will not define here a notion of stability for the didactic systems and we will accept to endow it with an equivalent meaning to "resist the disturbances". These disturbances are numerous, potentially as well as effectively, and their characteristics depend on the system being observed. For instance, in a classroom, some mistakes made by the pupils are going to request corrections only appealing to a simple individual treatment for which the teacher intervenes with only one pupil, and this will not be an important disruptive factor for the didactic system. On the other hand, some other, heavier or more significant mistakes as regards to the past or the present of teaching, need much more important interventions involving among others, injunctions to re-learn a lesson, explanations, complementary exercises... In other cases, the didactic system in its whole is reacting, integrating or rejecting a curricular modification, which is a matter for a similar phenomenon. G. Brousseau (1996) brought to the fore the fundamental structural role of regulation, in particular about the understanding of the teachers' role. It's a fundamental characteristic of his activity. G. Brousseau proposed an aid to study the regulation's instruments and the teacher's strategies : the changing of contract. Hence we can consider a typology "a priori" of different communication contracts whose stakes are knowledge, some of them being far beyond the mere communication. The principle of classification of these contracts consists in identifying the respective roles and responsibilities of the transmitter (in general the teacher) and the receiver (in general the learner) concerning the devolution of the exercise of an operating responsibility towards knowledge. The dynamics of teaching-learning is managed by changes of contract occurring during the process.

The second series of questions is linked to the structuration of the didactic system, in the sense of a dynamic sharing of sets of problems, activities, terms and symbolic forms. This listing is not complete and the whole cannot strictly speaking stand for a definition. To be operating from a didactic point of view, these sets of problems have got internal and external properties, enabling them to ensure their function, which is to allow teaching and learning. Following Y. Chevallard (1996) we can designate these sets as organizations.

Some of these properties have an epistemological dominant characteristic. They contribute to ensuring the appropriateness of knowledge and learnings, whose examined set allows their construction, with elements of mathematics in their double reference to the scholarly learnings and to the reference practices. In this scope, we can feel interest for the various environments in which some questions, some problems and therefore some objects, knowledge and techniques will be called for and contribute to founding the meaning of the aimed knowledge. Hence for example the notion and the problem of scales enable to train to work about the notion of ratio in numerical domains where one element of the ratio is a large number.

Other properties have a didactic dominant characteristic. On this account, let's quote the rate of use of some techniques, the order of appearance of problems and exercises, the representivness of some subsets of problems, the resumptions (of terms, of techniques, of problems). This listing is obviously not exhaustive and numerous studies are still necessary to ensure good identification of these functionally relevant properties of these sets. We must underline that the proposal worded here is not made without knowing the learnings many participants of the educative system might have developed during their practices about these sets of exercises (among others, teachers themselves and scholarbooks writers).

The third series of questions will call for the development of new and specific means of study, as much on the theoretical level as on the empirical one, the one of confrontation with contingency. It's a matter of studying, from a double view point of modelling and determining scopes and means for action, the connections between the effective school plans (as defined by constraints of the system which are specific neither to mathematics nor to strictly speaking didactics, but referring to general determiners of school organization) and the didactic systems which can, and have to, take place inside these plans. Three series of considerations bring these questions as a current issue.

The first one is linked to the progresses brought by the results of research in the field of mathematics teaching, and among others the one produced by the thematizations having a didactical characteristics (theory of situations and anthropological approach). Innocently speaking, the functioning of elements stemmed from research in a standard school context need to have them undergo specific adaptations even if the can rely on teachers' know-how.

The second series of considerations is linked to the recent diversification of the information media and the auxiliaries of the mathematical activity which are the mathematical softwares (Derive, Maple, Cabri Geometre) and the pocket calculators. They lead to evolutions in the domain of teaching and learning devices, but these evolutions remain restrained by the putting into practice of the didactical intentionality, founding and characterizing the teaching and learning systems at a scale in keeping with the scopes relating to these systems. This didactical intentionality is the one which gives structure to the didactic systems and determine their invariants and their regularities.

The third series of considerations is associated to the new requirements of teachers training, being an initial training or an in-house training. All of us have got a practical knowledge of the difficulties any participant (trainer or trained) operating inside our training systems might encounter, as it is an important part of our professional activity.

Instead of keeping in sterile confrontation such as theory versus practice, still cluttering too much our ways of thinking in this domain, it would be proper to substitute ways to generate the action of teaching and its relation to the didactics, bearing a real operatory value and being able to fit in with both the brief period of time of the initial learning and the longer periods of time of the in-house training; It is a matter to the study of which research can bring an amply significant contribution through adapted problematization.

5. Conclusion

In this paper we choose to took at things from a general point of view to sustain the debate which should first find its place in a set of themes dealing with the relationship between theoretical problematics and questions about teaching. From this view point, it seems more fruitful to detail a few central questions so that the issue of their relevance and their signification in the context of research in mathematic teaching could be brought up for discussion. Some of these questions already are being studied, set from the point of view provided by the theory of situations. Some others still don't have completely found this place allowing them to be studied from this viewpoint. The work this paper calls for might produce significant evolutions about relevant objects and their mutual relations. Whatever the case, this can only be positive as much for the questions themselves as for the chosen theoretical problematics.

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THE RESEARCH METHODS EMPLOYED BY CONTRIBUTORS TO THE PRAGUE DIDACTICS OF MATHEMATICS SEMINARS

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Abstract: *The article focuses on the methodology of research within the Prague Seminars of Didactics of Mathematics. Its main characteristics will be identified: the phenomenon of formalism as a research question, models of a cognitive net and their use for diagnosis, re-education and prevention as the main aims, introspection as a research method. Finally, a method of implementing research results will be presented.*
Keywords*: methodology, formalism, model.*

1. Introduction and Framework

In this article, we will identify and illustrate the main characteristics of research done within the Prague Seminars of Didactics of Mathematics led by M. Hejný. The research is focussed on the thinking processes and teaching/learning processes in mathematics. The majority of the research is of a qualitative nature and falls into what Schoenfeld (1991) calls basic research into cognition. “By basic in this context I mean work whose fundamental thrust is to inquire into the nature of thinking and learning processes, and which is not driven in any obvious way by applications.”

Our research consists of four main parts: modelling, diagnosis, re-education and prevention. Here only the first part will be presented. The following text will be written at three levels. Firstly, we will give a general description of the methodology. Secondly, the ideas presented in the general description will be illustrated by our research projects. Lastly, some references to the relevant work done by other contributors to the Prague Seminars will be given, especially in those parts of the methodology where our own research has not yet been developed. In the next section, we will present briefly the two research projects with which the main points will be illustrated.

2. Research Projects

2.1 A Student's Solution to a Word Problem

- see Stehlikova, 1996, 1995a, 1995c,d -

The initial goal of the research was to analyse a student's ability to solve word problems mainly from the point of view of formalism (see part 3). A group of five word problems was given to 11-year old children in five schools in the Czech Republic and three schools in England. The method of atomic analysis was chosen to get deeper insights into the students solving processes. The method was developed at the Bratislava Seminars in Mathematical Education and is described in PhD thesis (Stehlikova 1995a) and in (Stehlikova 1995c) in detail. Atomic analysis is based on two ideas, atomisation of the solving process and comparative analysis. It can be characterised as a method whose aim is to investigate the student's intellectual processes, i.e. the sequence of written ideas and mental steps which caused this sequence and the causality of this mechanism. The decisive role is played by cognitive and personal phenomena by means of which the thinking process is described. The analysis of a student's written solution is done with respect to one or more phenomena "atom by atom" (i.e. small meaningful parts of the solution). The elaboration of atomic analysis from the point of view of methodology is one of the most important results of the research (see Stehlikova 1995a).

2.2 Ability to Construct a Mental Representation of a Structure

- see Kubinova et al., 1997; Stehlikova, 1997 -

It is well known that some students are able to create their own pictures of structures for different parts of mathematics. The development of mathematics thinking in these cases is not based on the growth of the structure but mainly on re-structuring of their mathematical knowledge. On the other hand, other students only accumulate isolated pieces of knowledge. Such knowledge suffers from the “disease” of formalism (see part 3). In order to remove formal understanding, the mechanisms for the process of structuring must be discovered. The research project is therefore focussed on a student’s ability to discover and build a structure. The question is how does a structure grow and how the ability to recognise certain classifying characteristics in a system and to judge their importance, is developed. Phenomena describing the process of structuring are looked for.

As a research tool a non-standard arithmetic structure A_2 which is new to a student but which can be built in parallel with a structure the student already knows (Z as the ring without zero divisors) is being used. In this way we aimed to minimise the chance that a student’s previous knowledge will influence the research results. Concepts in the structure of A_2 are of two types: expected, i.e. the same or very similar to those in Z, and surprising, i.e. different from Z, which contribute to a student’s motivation and penetration of the structure.

Structure $A_2=(A_2,\oplus,\otimes)$ consists of the set $A_2=\{1,2,...,99\}$ of 99 natural numbers and two binary operations *z-addition* $\oplus$ and *z-multiplication* $\otimes$ defined as follows: $x\oplus y = R(x+y)$ and $x\otimes y = R(x\cdot y)$ where $R:N\rightarrow N$ we call a *reduction mapping*. The reduction mapping can be simply presented on the set of all three digit numbers ABC and four digit numbers ABCD as $R(100A+10B+C)=A+(10B+C)$, $R(1000A+100B+10C+D) =(10A+B)+(10C+D)$. For example $73\oplus 69=R(142)=1+42=43$, $81\otimes 90=R(R(7290))= R(72+90)=R(162)=1+62=63$.

3. Research Questions

Research questions may originate from schools or from previous research as “the second most significant products [of research] are stimuli to other researchers and teachers to test our conjectures for themselves in their own context” (Mason, 1998). In the Prague Seminars, the former is often the case. Central to its research is the phenomenon of formalism which “... is the most serious ‘disease’ of a student’s mathematical knowledge” (Hejny et al. 1990). *Formal knowledge* can be characterised in brief as follows: (1) Individual pieces of mathematical knowledge are weakly connected or even disconnected in a person's mind. They are stored in the memory as more or less isolated data. Connections among them are poor. A student is not able to use his/her knowledge in a different context, for instance for characterising other pieces of knowledge. (2) Mathematical knowledge is not connected with real life, i.e. it is not "processed" through the student's experience. For instance, a student is not able to find a model of a mathematical concept in real life. (3) A student's mathematical thinking is oriented towards the question 'How?' rather than 'Why?'. He/she tends to solve a problem by applying an algorithm previously learnt.

We speak about formal knowledge when a student can give a definition, an algorithm or a theorem but does not understand it, nor can they apply it in a different context. For example, Peter is asked in a lesson on analytic geometry to decide if all rotations in plane form a group or not. He knows the definition of group and all the criteria a binary operation must satisfy but cannot apply it in the context of geometry. His knowledge of group theory suffers from the disease of formalism. He needs more experience with different models of groups. On the other hand, we do not speak about formalism when a new piece of knowledge is presented to a student. It only exists in isolation at first and only after gaining more experience is it accommodated in the existing structure. It is only at a developmental stage then, not the manifestation of formalism.

Formalism can attack individual pieces of knowledge but it can "spread like a disease". It can penetrate into to the level of abilities and create a learning strategy in which a student prefers learning by rote to learning with understanding. If formalism attacks inner beliefs, a student is convinced that he/she is not able to learn mathematics with understanding and that he/she can only master learning by heart. When presented with a task, he/she does not attempt to gain insight but rather tries to use any algorithms by chance.

From the application point of view, the main aim of our research is a *diagnosis*, *re-education* and *prevention* of formalism. This leads to the necessity to explore the anatomy of the process of gaining knowledge from the standpoint of a student's thinking processes and to create a *model*.

4. How to Create Models

In this part, we will illustrate the methodology of creating models through identifying *phenomena*, *trace* and *mechanisms*. Let us characterise what we mean by these terms. Even though it will be done separately for each term, the distinction among them may be blurred and they can, and usually are, investigated at the same time.

4.1 Phenomena

The study of the thinking process begins by seeking those elements which link this process (so far unknown) to our knowledge of a cognitive structure. These elements will be called phenomena. A phenomenon answers the question, "What?". What we should concentrate on? What is important in a solving process? The process of identifying and later describing phenomena is long and specific methods are needed. "The need to identify and describe various cognitive structures in all phases of construction suggests methods such as the clinical interview and prolonged observation, that permit us to make inferences about the structures that underlie behaviour" (Noddings, 1990).

Illustration: In research 2.1, students written work from a standardised experiment was taken as the basis and the method of atomic analysis was used for eliciting relevant phenomena. For some examples of phenomena in research 2.1 see (Stehlikova 1995a,c). In 2.2, interviews with students were given preference. In current research, hardly anyone would deny the merits of clinical interview (see Ginsburg, 1981, for a detailed elaboration of this) even though it has disadvantages. Here we will describe in more detail the research process in research 2.2 (it will be written in ich-form). The research consists of two parts — the preparatory stage (steps 1 – 4) and the research itself (steps 5 – 11).

1. I, as an experimenter, discovered A2 through a set of problems. This process was recorded for future use.
2. I prepared a set of problems for the interviews with students related to my experience with the structure. The equations were chosen to show a wide range of

possibilities, including equations with more than one solution, no solution and those with the neutral element 99.

3. I conducted and recorded the first interview with Mary. First I showed Mary the set A_2 as the set of the first 99 natural numbers. Then I introduced reduction using several concrete examples. After this I showed Mary the operation of z-addition and z-multiplication, first generally and then on a concrete example.
4. I transcribed and analysed the interview. I listed several mistakes I had made when conducting the interview.
5. I prepared a second set of problems on the basis of my experience with the first experiment. Here it is:

(a) Reduction: r(100)=1, r(224)=26, r(1020)=30, r(1326)=39, r(2899)=r(127)=28, r(33246)=r(81)=81.
Task: Choose several natural numbers and reduce them.

(b) Additive and multiplicative problems: 6⊕60, 99⊕35, ... 6⊗9, 4⊗48, ... Task: Think about a word problem for some of these problems.

Note: This task was first used as a good motivator for the student. But it has become clear that it is a good task for discovering a neutral element. Martina found it when formulating these word problems. “We have 6 apples and we get 60 more. How many apples do we have altogether?” ”We have 68 apples and we get 97 more.” (There comes the surprise, sometimes it is better not to get any more.) “We have 35 apples and we get 99 more”. (Nothing changes.)

c) Simple equations: x⊕6=92, 61⊕x=4,... 3⊗x=45, 50⊗x=5... More difficult equations: 3⊗x⊕2=83, 5⊗x⊕10=5.

Note: This was the preparation for the first interview. In fact, students needed more time to complete these tasks (with variations mentioned in the note below). Later they were given other tasks such as divisibility, powers, square roots, sequences, quadratic equations etc.

6. Interviews were conducted. Each interview lasted between 30 to 60 minutes. Each student came as many times as he/she wanted (after school). They are all future mathematics teachers.

Note: During the interviews with different students the need to alter the way of presenting problems, to change their order etc. according to the situation became clear. So finally, only the first interview was more or less the same but the next interviews differed for different students according to their progress.

7. All interviews were transcribed word by word including all pauses, interruptions etc.
8. Using my experience working in the structure A_2 I prepared a set of "important stages" (what was important for me, what caused problems). These stages were used as a basis for the first analysis of interviews (i.e. the analysis was done with respect to one "important stage" at a time while all others were suppressed). The first list was as follows: (a) *Reduction* — Does a student understand it? What mistakes does he/she make? Does he/she understand the inverse process, i.e. that is for example 15=r(114)=r(213)=...? How long is the line of reductions? (b) *Subtraction* — How to introduce and define subtraction? Is it possible to do it without the knowledge of algebra? (c) Did a student find out that some equations have *more solutions*? How? (d) Does a student use the same *strategy* of solving equations as the author? Does he/she solve all equations in the same way or not? (e) Did a student find that there exists a *neutral element*? How?
9. "Important stages" were determined for individual students. The first list was augmented.
10. On the basis of several lists of "important stages" the first list of phenomena was made (for the lack of space it will not be presented here). By phenomenon we mean something which plays an important role in the process of discovering the structure of A_2.
11. The second analysis from the point of view of the first list of phenomena was done. All new interviews were analysed from the point of view of this list. This produced a new list of phenomena and this is also being augmented as the research progresses. (The names of phenomena need further clarification, S1 = natural or whole numbers, i.e. the known structure (N or Z with addition and N or Z with multiplication), S2 = the structure of A_2 i.e. the new structure).

(I) *The connection of S1 and S2 on the level of objects*: (a) clarification of the basic elements, e.g. with what certainty a student works inside A_2, does he/she work with numbers bigger than 99 or not; (b) elements which have a different character in S1 and S2, e.g. a neutral element (z-zero); (c) elements which are not present in S1 or S2, e.g. divisors of zero in S2. (II) *The connection of S1 and S2 on the level of operations*: reduction, inverse reduction, z-addition, z-multiplication, z-division, z-subtraction. (III) *The connection of S1 and S2 on the level of strategies*, e.g. the strategy of solving linear equations. (IV) *The transfer of experience between S1 and S2*: (a) with a test and adjustment, e.g. a student tries the strategy of solving quadratic equations as known from S1 and if it does not work tries to modify it; (b) formal (without test), e.g. it does not occur to a student that there could be something wrong, he/she just uses the method he/she knows; (c) no transfer (a student seems not to use his/her experience from S1 in the situation where it seems evident). (V) *Autonomous work*, e.g. a student asks his/her own questions: Does there exist an equation with 9 roots? Does a quadratic equation have two solutions? Does there exist a general rule for solving simple linear equations which would enable me just to look at it and know the answer (e.g. when I know one solution, I can quickly give all other solutions)?

We believe that phenomena are necessary for model creation, moreover they also contribute to the specification of research and exemplification of research aims. In the case of research 2.2, which is quite broad, it is our aim to focus more narrowly on one or several phenomena which we hope will prove "profitable".

4.2 Trace

A *trace* answers the question, "How?" It is the description of a time sequence of mental steps (states). A trace can be either local (at most within 12 hours, but usually about 30 minutes) or global which provides long term information (at least a week but usually half a year or more).

Illustration: In 2.2, a "local" trace describes what happened during a period of one interview which usually lasted 30-45 minutes. A "global" trace is what we intend to do in the future, to describe the change of the ability to build a structure in at least two years. In addition, this ability will be investigated with different age groups. The technology of tracing was elaborated also in research 2.1 (the technology of recording a student's solving process from the point of view of both what was written down and 'inner speech') (see Stehlikova 1995a) and in (Kratochvilova, 1997) in combinatorial problems.

4.3 Mechanisms

While a trace describes what happened, a *mechanism* explains why it happened. It answers the question "Why?", i.e. it identifies causes of the items described in the trace. If we know the mechanisms of a student's solving style, we are able to predict how he/she will solve a problem and/or explain why it was solved in the way it was.

Illustration: One of the universal mechanisms which was discovered within the Bratislava Seminars and which serves as a vehicle in our research, is the mechanism of the process of gaining knowledge (motivation, the stage of separate models, the stage of a universal model, a piece of knowledge, its crystallisation, its automation) (see Hejny et al., 1990). In 2.2, we observed the mechanism of a student's attitude towards a mistake. Mary, having made a mistake, always looked for it in the original solution. On the contrary, Ann never did this. She preferred to look for other ways to solve the problem.

4.4 Models

By means of the three preceding terms a *model* can be formed. By a model, we mean a projection of a certain situation or process, which is hidden, to a context which visualises this situation or process. The model has to be quite universal. The question is how big a sample of concrete examples the model can describe and to what depth it goes. These two tendencies often go against each other. Today's trend is by case study. Fewer examples but having greater depth are preferred (see e.g. Schoenfeld, 1991).

Illustration: In research 2.1, we made a model - scheme based on the basic dichotomy of calculational and semantic levels of solving word problems (see Stehlikova 1995c,d). Through this model we are able to follow the thinking processes (traces) of a particular student, describe them by means of discovered phenomena and identify his/her difficulties. The research 2.2 is currently at the level of discovering phenomena and trace but in the long-term perspective, the goal is to create a model which will diagnose the ability to build a structure. The model will not be a simple one because the research is not focussed on a single concept but on the whole structure.

Other related projects: A model of grasping word problems can be found in (Hejny, 1995a). For a model of solving processes in word problems, mainly from the point of view of strategies, see (Novotna et al. 1994).

5. Inner Research

In the previous paragraphs it was shown how models are based on information gained through a student's written or oral work. But there is another important way of getting information and that is introspection as another characteristic of the methodology built within the Prague Seminars. In our opinion, our approach corresponds with inner research in Mason (1998) who claims that "inner research is about developing sensibility" and "I need to be sensitive to the structure, importance and techniques of a topic if I am to assist others to alter the structure of their attention". It is distinctive to the Prague Seminars that researchers (and teachers) go through the similar situations (usually at a higher level) as their students to get a deeper insight into a student's thinking processes (see also Littler et al. 1998).

Illustration: In research 2.2, the experimenter was first introduced into the structure of A_2 in a similar way to the students and could therefore presuppose many of the difficulties (obstacles) encountered by students in the process of their penetrating the structure. Moreover, through introspection a 'new' use of A_2 was discovered, namely how it can help produce a meaningful and natural introduction to the basic concepts of group algebra. It brought to the surface the question whether it is possible at the university level to teach the structure of algebra (or any other structure) so that the knowledge is not afflicted by formalism.

6. Implementing Research Results

There are vast numbers of articles written on implementing research results into practice. Here we would like to present two ways which are used within the Prague Seminars. The first way corresponds to the current trend in mathematics education. Practising teachers are encouraged to co-operate with researchers.

The second way will be illustrated in more detail. The following citation summarises nicely our approach. “In order to maximise the effectiveness of research effort in mathematics education it behoves us to reflect on what we go through when we undertake research, and to try to offer and support practitioners in undertaking similar or analogous personal reconstruction of research activities“ (Mason, 1998). The Department of Mathematics and the Didactics of Mathematics of the Faculty of Education, Charles University organised a whole series of workshops (called “Iniciativa“) in which practising teachers took part. The aim was to introduce and let them experience some research techniques rather than share the research results with them.

Our workshop focussed on the method of atomic analysis and its use for changing a teacher’s attitude towards assessment of their students’ work. We created a similar climate for teachers as we, the researchers, had when using atomic analysis. Rather than tell the teachers the basic principles of the method and its advantages for their practice we encouraged them to do their own experiments and to analyse the students’ outcomes. We wanted them to see for themselves what merits the method had and how they could use it in their teaching. The feedback from this project shows that for the majority of them, their sensitivity towards their students and especially their work, increased (Stehlikova, 1995b). For similar projects see (Hejny, 1995b; Koman et al. 1995; Kubinova, 1995; Novotna, 1995).

7. Conclusion

We have identified and illustrated some distinctive features of the methodology elaborated within the Prague Seminars. We have indicated directions *in* which our research project will proceed. Several other models are being created using the same methodology and it is our task to make an overall project analysis using the comparative method.

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Acknowledgment

The contribution was financially aided by the grant GACR No. 406/97/P132 "Structuring mathematical knowledge" and the grant GAUK No. 302/1998/A PP/PedF "The phenomena of the mathematical part of the cognitive structure". It was supported by the Ministry of Education of the Czech Republic (Project PG 98374).

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